

Kinematics of vector fields

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Abstract

Different (not only by sign) affine connections are introduced for contravariant and covariant tensor fields over a differentiable manifold by means of a non-canonical contraction operator, defining the notion dual space and commuting with the covariant and with the Lie-differential operator. Classification of the linear transports on the basis of the connections between the connections is given. Notion of relative velocity and relative acceleration for vector fields are determined. By means of these kinematic characteristics several other types of notions as shear velocity, shear acceleration, rotation velocity, rotation acceleration, expansion velocity and expansion acceleration are introduced and on their basis the auto-parallel vector fields are classified.

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1 Introduction

The evolution of the differential geometry is due to a great extent to ideas connected with attempts for describing different types of physical interactions by means of differential geometric methods. The created at the beginning of the 20-th century theory of relativity carried out the hypotheses of some geometers about connections between space-time and material systems, evaluating in it, as well as ideas of many physicists, trying to investigate mathematical models of physical systems by means of differential-geometric structures (Lichnerowicz 1979).

The evolution of the special and the general theory of relativity and the attempts for their generalization and connection with other theories of physical interactions provided opportunity for using new geometrical structures (different types of fiber bundles, geometries, different from the Riemannian geometry, complex manifolds, different basic vector fields and metric tensor fields, different connections) (Ivanenko, Pronin, Sardanashvily 1985), (Barvinskii, Ponomarev, Obukhov 1985), (Hehl 1966, 1970, 1973, 1974). Different methods are also used in finding solutions of equations for the gravitational field connected with differential-geometric structures over manifolds (special vector and tensor fields, spinor fields etc.). Problems, arising in solving the equations of modern gravitational theories, induced an evolution of new approaches to existing mathematical models and created preconditions for working out new differential-geometric methods (Kramer, Stephani, MacCallum, Herlt 1980), (Kramer, Stephani 1983)..

For a century only the mathematical models of the space-time went from the Euclidean, Minkowskian and (pseudo)Riemannian space-time to more sophisticated spaces with linear (affine) connection and metric (Hecht, Hehl 1991), (Hehl, von der Heyde 1973), (Hehl, Kerlik 1978). The generalization of the Newton's theory of gravitation in the Einstein's theory of gravitation (ETG) was an important step toward the use of two essential differential-geometric objects in the gravitational theory: the metric, which allows the definition of a distance between two points of a manifolds, considered as a model of space-time and the affine connection, which allows the transport of a geometric object from one point to another point of a manifold and a comparison of two objects at one and the same point. In the Riemannian geometry these two geometric objects are connected each other - the Levi-Civita (symmetric) connection can be given by means of the Riemannian metric. This was at the beginning

the mathematical basis for the ETG and its generalization in the range of the Riemannian geometry. But later on the generalizations went on two different directions: in the first one two different metrics over the one and the same manifold were introduced (bi-metric theory of gravitation (Rosen 1973, 1974), (Logunov, Mestvirishvili 1989)) and in the second - two different connections for the tensor fields over a manifold were introduced (bi-connection theory of gravitation (Tchernikov 1987, 1988, 1990)). For Riemannian spaces these two directions came one into another. In the last few years new attempts are made to revive the ideas of Weyl, (Edington 1925) and (Schroedinger 1950) for using manifolds with independent affine connection and metric (spaces with affine connection and metric or (L_n, g) -spaces) as a model of space-time in a theory of gravitation (Hecht, Hehl 1991). In such spaces the connection for co-tangent vector fields (as dual to the tangent vector fields) differs from the connection for the tangent vector fields only by sign. The last fact is due to the definition of dual vector spaces over points of a manifold, which is a trivial generalization of the definition of algebraic dual vector spaces from the multi linear algebra (Greub 1978), (Efimov, Rosendorn 1974), (Greub, Halperin, Vanstone 1972, 1973), (Bishop, Goldberg 1968), . The whole modern differential geometry is build on the one hand as a rigorous logical structure having as one of its main assumption the canonical definition for algebraic dual vector spaces (with equal dimensions) (Choquet-Bruhat, DeWitte-Morette, Dillard-Bleik 1977). On the other hand, the possibility of introducing a non-canonical definition for algebraic dual vector spaces (with equal dimensions) has been pointed out by many mathematicians (Kobayashi, Nomizu 1963) who have not exploited this possibility for further evolution of the differential-geometric structures and its applications. The canonical definition of dual spaces is so naturally embedded in the ground of the differential geometry that no need has occurred for changing it (Matsushima 1972), (Boothby 1975), (Lovelock, Rund 1975), (Norden 1976). But the last time evolution of the mathematical models for describing the gravitational interaction on classical level shows a tendency to generalizations using spaces with affine connection and metric, which can be also generalized using the freedom of the differential-geometric preconditions. The fact, that affine connection, which in a point or over a curve in Riemannian spaces can vanish (principle of equivalence in ETG), can also vanish under special choice of the basic system in a space with affine connection and metric (von der Heyde 1975), (Iliev 1992), shows that the equivalence principle in the ETG is only a corollary of the mathematical apparatus used in this theory. Therefore, every differentiable manifold with affine connection and metric can be used as a model for space-time in which the equivalence principle holds. But, if the manifold has two different (not only by sign) connections for tangent and co-tangent vector fields, the situation changes and is worth being investigated.

In Section 1. the notions contravariant and covariant affine connection are defined for contravariant and covariant tensor fields over differentiable manifolds. It is shown that these two different (not only by sign) connections can be introduced by means of changing the canonical definition of dual vector spaces (respectively of dual vector fields).

In Einstein's theory of gravitation (ETG) kinematic notions related to the notion relative velocity such as shear velocity tensor (shear velocity, shear) σ , rotation velocity tensor (rotation velocity, rotation) ω and expansion velocity (expansion) θ , are used in finding solutions of special types of Einstein's field equations and in the description of the properties of the (pseudo)Riemannian spaces without torsion (V_n -spaces). By means of these notions a classification of V_n -spaces, admitting special types of geodesic vector fields has been proposed (Ehlers 1961). The same kinematic characteristics are also necessary for description of the projections of the Riemannian (curvature) tensor and the Ricci tensor along a non-isotropic (non-null) vector field (Kramer, Stephani, MacCallum, Herlt 1980) and in obtaining and using the Raychaudhuri identity (Hawking, Ellis 1973) in V_n -spaces.

The kinematic characteristics, connected with the notion relative velocity can be generalized for vector fields over differentiable manifolds with contravariant and covariant affine connection and metric $((\bar{L}_n, g)$ -spaces) so that in the case of (L_n, g) - and V_n -spaces (as a special case of (\bar{L}_n, g) -spaces) and for normalized non-isotropic vector fields these characteristics are the same as those, introduced in the ETG. In analogous way as in the case of the kinematic characteristics, related to the notion of relative velocity, it is possible to introduce kinematic characteristics, related to the notion of relative acceleration such as shear acceleration tensor (shear acceleration), rotation acceleration tensor (rotation acceleration) and expansion acceleration (Manoff 1985, 1992).

In Section 2., 3. and 4. the corresponding for (\bar{L}_n, g) -spaces notions of relative velocity and relative acceleration are introduced. By means of these kinematic characteristics several other types of notions such as shear velocity, shear acceleration, rotation velocity, rotation acceleration, expansion velocity and expansion acceleration are investigated and on their basis the auto-parallel vector fields in (\bar{L}_n, g) -spaces are classified. The generalizations compared with those in (L_n, g) -spaces (differentiable manifolds with affine connection and metric) appear only in the explicit forms of the expressions, written in a corresponding basis (or in other words - only in index forms).

On the basis of the introduced notions deviation equations (playing important role in gravitational physics) and Lagrangian theories of tensor fields can be considered in (\bar{L}_n, g) -spaces. If kinematic characteristics of a dynamic system are given as preconditions, then the corresponding type of differentiable manifold (which allows such characteristics) can be chosen as a model of space-time, where the evolution of the system is taking place. This idea connects different differential-geometric structures used for describing physical systems on classical level. The main objects taken in such type of considerations can be given schematically as follows

Differentiable manifolds with contravariant and covariant affine connections and metric
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Kinematic characteristics of contravariant vector fields

Deviation equations for vector fields
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Lagrangian theory of tensor fields

= . =

In the present paper we will concentrate our attention only on the kinematic characteristics of contravariant vector fields but some main ideas and definitions will be outlined.

2 Differentiable manifolds with contravariant and covariant affine connection and metric $[(\bar{L}_n, g)$ -spaces]

The notion algebraic dual vector space can be introduced in such a way (Efimov, Rosendorn 1974), in which the two vector spaces (the considered and its dual vector space) are two independent (finite) vector spaces with equal dimensions.

2.1 Contraction operator

Let X and X^* be two vector spaces with equal dimensions $\dim X = \dim X^* = n$. Let S be an operator (mapping) such that to every pair of elements $u \in X$ and $p \in X^*$ sets an element of the field K (R or C), i.e.

$$S : (u, p) \rightarrow z \in K, u \in X, p \in X^* .$$

Definition 1. The operator (mapping) S is called *contraction operator* S , if it is a bi-linear symmetric mapping, i.e. if it fulfills the following conditions:

- a) $S(u, p_1 + p_2) = S(u, p_1) + S(u, p_2)$, $\forall u \in X$, $\forall p_i \in X^*$, $i = 1, 2$,
- b) $S(u_1 + u_2, p) = S(u_1, p) + S(u_2, p)$, $\forall u_i \in X$, $i = 1, 2$, $\forall p \in X^*$,
- c) $S(\alpha u, p) = S(u, \alpha p) = \alpha \cdot S(u, p)$, $\alpha \in K$,
- d) non-degeneracy: if u_1, \dots, u_n are linear independent in X and $S(u_1, p) = 0, \dots, S(u_n, p) = 0$, then the p is the null element in X^* . In analogous way, if p_1, \dots, p_n are linear independent in X^* and $S(u, p_1) = 0, \dots, S(u, p_n) = 0$, then u is the null element in X ,
- e) symmetry: $S(u, p) = S(p, u)$, $\forall u \in X$, $\forall p \in X^*$.

Let e_1, \dots, e_n be an arbitrary basis in X , and let e^1, \dots, e^n be an arbitrary basis in X^* . Let $u = u^i \cdot e_i \in X$ and $p = p_k \cdot e^k \in X^*$.

From the properties a) and b) it follows that

$$S(u, p) = f^k_{\ i} \cdot u^i \cdot p_k = u^i \cdot p_{\bar{i}} = p_k \cdot u^{\bar{k}} \ , \ p_{\bar{i}} = f^k_{\ i} \cdot p_k \ , \ u^{\bar{k}} = f^k_{\ i} \cdot u^i \ ,$$

where

$$f^k_{\ i} = S(e_i, e^k) = S(e^k, e_i) \in K \ .$$

In this way, the result of the action of the contraction operator S is expressed in terms of a bi-linear form. The property non-degeneracy d) means the non-degeneracy of the bi-linear form. The result $S(u, p)$ can be defined in different ways by giving arbitrary numbers $f^k_{\ i} \in K$, for which the condition $\det(f^k_{\ i}) \neq 0$ and the conditions a) - d) are fulfilled.

Remark. In the canonical approach $S = C$ and $C(e_i, e^k) = C(e^k, e_i) = g^k_i$, $g^k_i = 1$ for $k = i$, $g^k_i = 0$ for $i \neq k$. The contraction operator C is the corresponding to the canonical approach mapping (Boothby 1975), (Matsushima 1972)

$$C(u, p) = C(p, u) = p(u) = p_i \cdot u^i \ .$$

Definition 2. (Mutually) *algebraic dual vector spaces* := The spaces X and X^* are called (mutually) dual spaces, if an contraction operator acting on them is given and they are considered together with this operator (i.e. (X, X^*, S) with $\dim X = n = \dim X^*$ defines the two (mutually) dual vector spaces X and X^*).

Remark. The generalization of the notion of algebraic dual vector spaces for the case of vector fields over differentiable manifold is a trivial one. The vector fields are considered as sections of vector bundles over a manifold. The vector bases become dependent on the points of the manifold and the numbers $f^i_{\ j}$ are considered as functions over the manifold.

Vector and tensor fields over a differentiable manifold are provided with the structure of a linear (vector) space by defining the corresponding operations at every point of the manifold.

Thus, the definition of algebraic dual vector spaces over manifolds by means of the contraction operator S as a generalization of the contraction operator C allows considerations including functions $f^i_{\ j} \in C^k(M)$ instead of the Kroneker symbol g^i_j .

2.2 Covariant differential operator. Contravariant and covariant affine connection

The notion affine connection can be defined in different ways but in all definitions a linear mapping is given, which to a given vector of a vector space over a point x of a manifold M juxtaposes a corresponding vector from the same vector space in this point. The corresponding vector is identified as vector of the vector space over another point of the manifold M . The way of identification is called *transport* from a point to another point of the manifold.

Definition 3. *Affine connection* over a differentiable manifold M . Let $V(M)$ ($\dim M = n$) be the set of all (smooth) vector fields over the manifold M . The mapping

$$\nabla : V(M) \times V(M) \rightarrow V(M) ,$$

by means of

$$\nabla(u, v) \rightarrow \nabla_u v , u, v \in V(M) ,$$

with the following properties

- a) $\nabla_u(v + w) = \nabla_u v + \nabla_u w$, $u, v, w \in V(M)$,
- b) $\nabla_u(fv) = (uf).v + f.\nabla_u v$, $f \in C^r(M)$, $r \geq 1$,
- c) $\nabla_{u+v}w = \nabla_u w + \nabla_v w$,
- d) $\nabla_{fu}v = f.\nabla_u v$,

is called *affine connection* over the manifold M .

Definition 4. *Covariant differential operator*. The linear differential operator (mapping) ∇_u with the following properties

- a) $\nabla_u(v + w) = \nabla_u v + \nabla_u w$, $u, v, w \in V(M)$,
- b) $\nabla_u(fv) = (uf).v + f.\nabla_u v$, $f \in C^r(M)$, $r \geq 1$,
- c) $\nabla_{u+v}w = \nabla_u w + \nabla_v w$,
- d) $\nabla_{fu}v = f.\nabla_u v$,
- e) $\nabla_u f = uf$, $f \in C^r(M)$, $r \geq 1$,
- f) $\nabla_u(v \otimes w) = \nabla_u v \otimes w + v \otimes \nabla_u w$ (Leibnitz rule), \otimes is the sign for tensor product,

is called *covariant differential operator along the vector u* .

The result $\nabla_u v$ of the action of the covariant differential operator on v is often called *covariant derivative of the vector field v* along the vector field u .

In a given chart (co-ordinate system) the determination of $\nabla_{e_\alpha} e_\beta$ in the basis $\{e_\alpha\}$ defines the components $\nabla_{\beta\gamma}^\alpha$ of the affine connection ∇

$$\nabla_{e_\alpha} e_\beta = \nabla_{\alpha\beta}^\gamma . e_\gamma , \alpha, \beta, \gamma = 1, \dots, n .$$

Definition 5. *Space with affine connection*. Differentiable manifold M , provided with affine connection ∇ , i.e. the pair (M, ∇) , is called space with affine connection.

The action of the covariant differential operator on a contravariant (tangential) co-ordinate basic vector field ∂_i over M along another contravariant co-ordinate basic vector field is determined by the affine connection $\nabla = \Gamma$ with the components Γ_{ij}^k in a given chart (co-ordinate system) defined through

$$\nabla_{\partial_j} \partial_i = \Gamma_{ij}^k . \partial_k .$$

For a non-co-ordinate contravariant basis

$$e_\alpha \text{ (or } e_i) \in T(M), T(M) = \cup_{x \in M} T_x(M)$$

$$\nabla_{e_\beta} e_\alpha = \Gamma_{\alpha\beta}^\gamma . e_\gamma .$$

Definition 6. *Contravariant affine connection.* Affine connection $\nabla = \Gamma$ induced by the action of the covariant differential operator on contravariant vector fields is called contravariant affine connection.

The action of the covariant differential operator on a covariant (dual to contravariant) basic vector field e^α ($e^\alpha \in T^*(M)$, $T^*(M) = \cup_{x \in M} T_x^*(M)$) along a contravariant basic (non-co-ordinate) vector field e_β is determined by affine connection $\nabla = P$ with components $P_{\beta\gamma}^\alpha$ defined through

$$\nabla_{e_\beta} e^\alpha = P_{\beta\gamma}^\alpha \cdot e^\gamma .$$

For a co-ordinate basis dx^i

$$\nabla_{\partial_j} dx^i = P_{kj}^i \cdot dx^k .$$

Definition 7. *Covariant affine connection.* Affine connection $\nabla = P$ induced by action of the covariant differential operator on covariant vector fields is called covariant affine connection.

Definition 8. *Space with contravariant and covariant affine connection* (\bar{L}_n -space). Differentiable manifold provided with contravariant affine connection Γ and covariant affine connection P is called space with contravariant and covariant affine connection.

Definition 9. *Space with contravariant and covariant affine connection, and metric* ((\bar{L}_n, g) -space). Differentiable manifold provided with contravariant affine connection Γ and covariant affine connection P , and metric g is called space with contravariant and covariant affine connection and metric.

The connection between the two connections Γ and P is based on the connection between the two dual spaces $T(M)$ and $T^*(M)$, which on its side is based on the existence of the contraction operator S . Usually commutation relations are required between the contraction operator and the covariant differential operator in the form

$$S \circ \nabla_u = \nabla_u \circ S .$$

If the last operator equality in the form $\nabla_{e_\gamma} \circ S = S \circ \nabla_{e_\gamma}$ is used for acting on the tensor product $e^\alpha \otimes e_\beta$ of two basic vector fields $e^\alpha \in T^*(M)$ and $e_\beta \in T(M)$, then

$$\nabla_{e_\gamma} (S(e^\alpha \otimes e_\beta)) = S(\nabla_{e_\gamma} (e^\alpha \otimes e_\beta)) ,$$

and the relation follows

$$e_\gamma f^\alpha{}_\beta = \Gamma_{\beta\gamma}^\delta \cdot f^\alpha{}_\delta + P_{\delta\gamma}^\alpha \cdot f^\delta{}_\beta ,$$

(in a non-co-ordinate basis)

or

$$f^i{}_{j,k} = \Gamma_{jk}^l \cdot f^i{}_l + P_{lk}^i \cdot f^l{}_j ,$$

$$f^i{}_{j,k} = \partial_k f^i{}_j ,$$

(in a co-ordinate basis).

The last equality can be considered from two different points of view:

1. If $P_{jk}^i(x^l)$ and $\Gamma_{jk}^i(x^l)$ are given functions of co-ordinates in M , then the equality appears as a system of equations for the unknown functions $f_{\cdot j}^i(x^k)$. The solutions of these equations determine the action of the contraction operator S on the basic vector fields for given components of both the connections. The integrability conditions for the equations can be written in the form

$$R^m_{\cdot jkl} \cdot f^i_{\cdot m} + P^i_{\cdot mkl} \cdot f^m_{\cdot j} = 0 ,$$

where $R^m_{\cdot jkl}$ are the components of the contravariant curvature tensor, constructed by means of the contravariant affine connection Γ , and $P^i_{\cdot mkl}$ are the components of the covariant curvature tensor, constructed by means of the covariant affine connection P , where $[R(\partial_i, \partial_j)]dx^k = P^k_{\cdot lij} \cdot dx^l$, $[R(\partial_i, \partial_j)]\partial_k = R^l_{\cdot kij} \cdot \partial_l$, $R(\partial_i, \partial_j) = \nabla_{\partial_i} \nabla_{\partial_j} - \nabla_{\partial_j} \nabla_{\partial_i}$.

2. If $f^i_{\cdot j}(x^l)$ are given as functions of the co-ordinates in M , then the conditions for $f^i_{\cdot j}$ determine the connection between the components of the contravariant affine connection Γ and the components of the covariant affine connection P on the ground of the predetermined action of the contraction operator S on basic vector fields.

If $S = C$, i.e. $f^i_{\cdot j} = g_j^i$, then the conditions for $f^i_{\cdot j}$ are fulfilled for every $P = -\Gamma$, i.e.

$$P_{jk}^i = -\Gamma_{jk}^i .$$

This fact can be formulated as the following proposition:

Proposition 1. $S = C$ is a sufficient condition for $P = -\Gamma$ ($P_{jk}^i = -\Gamma_{jk}^i$).

Corollary. If $P \neq -\Gamma$, then $S \neq C$, i.e. if the covariant affine connection P has to be different from the contravariant affine connection Γ not only by sign, then the contraction operator S has to be different from the canonical contraction operator C (if S commutes with the covariant differential operator).

The corollary allows introduction of different (not only by sign) contravariant and covariant connection by using contraction operator S , different from the canonical operator C .

Example. If $f^i_{\cdot j} = \varphi \cdot g_j^i$, where $\varphi \in C^r(M)$, $\varphi \neq 0$, then $P_{jk}^i = -\Gamma_{jk}^i + (\log \varphi)_{\cdot k} \cdot g_j^i$.

The covariant derivatives of contravariant vector fields can be written in an arbitrary co-ordinate or non-co-ordinate basis

$$\begin{aligned} \nabla_u v &= (v^i_{\cdot j} + \Gamma_{kj}^i v^k) u^j \cdot \partial_i = v^i_{\cdot j} u^j \cdot \partial_i , \quad u, v \in T(M) , \\ &\quad \text{(in a co-ordinate basis),} \\ \nabla_u v &= (e_\beta v^\alpha + \Gamma_{\gamma\beta}^\alpha \cdot v^\gamma) u^\beta \cdot e_\alpha = v^\alpha_{\cdot \beta} u^\beta \cdot e_\alpha , \\ \nabla_u v &= (e_j v^i + \Gamma_{kj}^i v^k) u^j \cdot e_i = v^i_{\cdot j} u^j \cdot e_i , \\ &\quad \text{(in a non-co-ordinate basis with different type of indices).} \end{aligned}$$

In analogous way the covariant derivative of covariant vector fields can be

written in an arbitrary co-ordinate or non-co-ordinate basis

$$\begin{aligned}\nabla_u p &= (p_{i,j} + P_{ij}^k p_k) u^j . dx^i = p_{i;j} u^j . dx^i, \quad p \in T^*(M), \quad u \in T(M), \\ &\quad \text{(in a co-ordinate basis),} \\ \nabla_u p &= (e_\beta p_\alpha + P_{\alpha\beta}^\gamma p_\gamma) u^\beta . e^\alpha = p_{\alpha/\beta} u^\beta . e^\alpha, \\ \nabla_u p &= (e_j p_i + P_{ij}^k p_k) u^j . e^i = p_{i/j} u^j . e^i, \\ &\quad \text{(in a non-co-ordinate basis with different type of indices).}\end{aligned}$$

The action of the covariant differential operator on contravariant and covariant tensor fields as well as on mixed tensor fields with rank $\succ 1$ is generalized in trivial manner on the ground of the Leibnitz rule, which holds for this operator.

If the Kroneker tensor is defined in the form

$$Kr = g_j^i . \partial_i \otimes dx^j = g_\beta^\alpha . e_\alpha \otimes e^\beta,$$

then the components of the contravariant and covariant affine connection differ from each other by the components of the covariant derivative of the Kroneker tensor, i.e.

$$\begin{aligned}\Gamma_{jk}^i + P_{jk}^i &= g_{j;k}^i, \\ \Gamma_{\beta\gamma}^\alpha + P_{\beta\gamma}^\alpha &= g_{\beta/\gamma}^\alpha.\end{aligned}$$

Remark. In the special case, when $S = C$, and in the canonical approach $g_{j;k}^i = 0$ ($g_{\beta/\gamma}^\alpha = 0$).

2.3 Lie-differential operator

The Lie-differential operator \mathcal{L}_ξ along the contravariant vector field ξ appears as another operator, which can be constructed by means of contravariant vector field. His definition can be considered as a generalization of the notion Lie derivative of tensor fields (Slebodzinski 1931), (Yano 1957), (Kobayashi, Nomizu 1963), (Lightman, Press, Price, Teukolsky 1975).

Definition 10. $\mathcal{L}_\xi :=$ Lie-differential operator along the contravariant vector field ξ with the following properties:

- a) $\mathcal{L}_\xi : V \rightarrow \overline{V} = \mathcal{L}_\xi V$, $V, \overline{V} \in \otimes^l(M)$,
- b) $\mathcal{L}_\xi : W \rightarrow \overline{W} = \mathcal{L}_\xi W$, $W, \overline{W} \in \otimes_k(M)$,
- c) $\mathcal{L}_\xi : K \rightarrow \overline{K} = \mathcal{L}_\xi K$, $K, \overline{K} \in \otimes^l_k(M)$,
- d) linear operator with respect to tensor fields,
 $\mathcal{L}_\xi(\alpha.V_1 + \beta.V_2) = \alpha.\mathcal{L}_\xi V_1 + \beta.\mathcal{L}_\xi V_2$, $\alpha, \beta \in F(R \text{ or } C)$, $V_i \in \otimes^l(M)$,
 $i = 1, 2$,
 $\mathcal{L}_\xi(\alpha.W_1 + \beta.W_2) = \alpha.\mathcal{L}_\xi W_1 + \beta.\mathcal{L}_\xi W_2$, $W_i \in \otimes_k(M)$, $i = 1, 2$,
 $\mathcal{L}_\xi(\alpha.K_1 + \beta.K_2) = \alpha.\mathcal{L}_\xi K_1 + \beta.\mathcal{L}_\xi K_2$, $K_i \in \otimes^l_k(M)$, $i = 1, 2$,
- e) linear operator with respect to the contravariant field ξ ,
 $\mathcal{L}_{\alpha\xi + \beta u} = \alpha.\mathcal{L}_\xi + \beta.\mathcal{L}_u$, $\alpha, \beta \in F(R \text{ or } C)$, $\xi, u \in T(M)$,
- f) differential operator, obeying the Leibnitz rule,
 $\mathcal{L}_\xi(S \otimes U) = \mathcal{L}_\xi S \otimes U + S \otimes \mathcal{L}_\xi U$, $S \in \otimes^m_q(M)$, $U \in \otimes^k_l(M)$,
- g) action on function $f \in C^r(M)$, $r \geq 1$,
 $\mathcal{L}_\xi f = \xi f$, $\xi \in T(M)$,

h) action on contravariant vector field,
 $\mathcal{L}_\xi u = [\xi, u]$, $\xi, u \in T(M)$, $[\xi, u] = \xi \circ u - u \circ \xi$,
 $\mathcal{L}_\xi e_\alpha = [\xi, e_\alpha] = -(e_\beta \xi^\alpha - \xi^\gamma C_{\gamma\beta}{}^\alpha) e_\alpha$,
 $\mathcal{L}_{e_\alpha} e_\beta = [e_\alpha, e_\beta] = C_{\alpha\beta}{}^\gamma e_\gamma$, $C_{\alpha\beta}{}^\gamma \in C^r(M)$,
 $\mathcal{L}_\xi \partial_i = -\xi^j{}_{,i} \partial_j$, $\mathcal{L}_{\partial_i} \partial_j = [\partial_i, \partial_j] = 0$,
i) action on covariant basic vector field,
 $\mathcal{L}_\xi e^\alpha = k^\alpha{}_\beta(\xi) e^\beta$, $\mathcal{L}_{e_\gamma} e^\alpha = k^\alpha{}_{\beta\gamma} e^\beta$,
 $\mathcal{L}_\xi dx^i = k^i{}_j(\xi) dx^j$, $\mathcal{L}_{\partial_k} dx^i = k^i{}_{jk} dx^j$.

The action of the Lie-differential operator on covariant basic vector field is determined by its action on contravariant basic vector field and the commutation relations between the Lie-differential operator and the contraction operator S .

2.3.1 Lie derivative of covariant co-ordinate basic vector fields

The commutation relations between the Lie-differential operator \mathcal{L}_ξ and the contraction operator S in the case of basic co-ordinate vector fields can be written in the form

$$\begin{aligned}\mathcal{L}_\xi \circ S(dx^i \otimes \partial_j) &= S \circ \mathcal{L}_\xi(dx^i \otimes \partial_j) , \\ \mathcal{L}_\xi \circ S(e^\alpha \otimes e_\beta) &= S \circ \mathcal{L}_\xi(e^\alpha \otimes e_\beta) ,\end{aligned}\tag{1}$$

where

$$\begin{aligned}\mathcal{L}_\xi \circ S(dx^i \otimes \partial_j) &= \xi f^i{}_j = f^i{}_{j,k} \xi^k , \\ S \circ \mathcal{L}_\xi(dx^i \otimes \partial_j) &= S \circ (\mathcal{L}_\xi dx^i \otimes \partial_j + dx^i \otimes \mathcal{L}_\xi \partial_j) = \\ &= S(\mathcal{L}_\xi dx^i \otimes \partial_j) + S(dx^i \otimes \mathcal{L}_\xi \partial_j) .\end{aligned}\tag{2}$$

Since $\mathcal{L}_\xi dx^i = k^i{}_j(\xi) dx^j$, $\mathcal{L}_{\partial_k} dx^i = k^i{}_{jk} dx^j$, where $k^i{}_j(\xi) \in C^r(M)$, $k^i{}_{jk} \in C^r(M)$, $k^i{}_j(\xi)$ and $k^i{}_{jk}$ have to be determined by means of the commutation relations between \mathcal{L}_ξ and S and their action on dx^i and ∂_j (respectively e^α and e_β) on the basis of the relations

$$\begin{aligned}\mathcal{L}_\xi \partial_j &= -\xi^k{}_{,j} \partial_k , \quad S(dx^i \otimes \mathcal{L}_\xi \partial_j) = -\xi^k{}_{,j} f^i{}_k , \\ S(\mathcal{L}_\xi dx^i \otimes \partial_j) &= k^i{}_l(\xi) f^l{}_j , \\ S[\mathcal{L}_\xi(dx^i \otimes \partial_j)] &= k^i{}_l(\xi) f^l{}_j - \xi^k{}_{,j} f^i{}_k = \\ &= \mathcal{L}_\xi[S(dx^i \otimes \partial_j)] = f^i{}_{j,k} \xi^k .\end{aligned}\tag{3}$$

From the last expression the condition follows for $k^i{}_l(\xi)$

$$k^i{}_l(\xi) f^l{}_j = \xi^k{}_{,j} f^i{}_k + f^i{}_{j,k} \xi^k .\tag{4}$$

By means of the non-degenerate inverse matrix $(f^i{}_j)^{-1} = (f_j{}^i)$ and the connections $f^i{}_k f_j{}^k = g_j^i$, $f^k{}_i f_k{}^j = g_i^j$, after multiplication of the equality for $k^i{}_l(\xi)$ with $f_m{}^j$ and summation over j , the explicit form for $k^i{}_j(\xi)$ is obtained in the form

$$k^i{}_j(\xi) = f_j{}^l \xi^k{}_{,l} f^i{}_k + f_j{}^l f^i{}_{l,k} \xi^k .\tag{5}$$

For $\mathcal{L}_{\partial_k} dx^i = k^i_{\ j}(\partial_k).dx^j = k^i_{\ jk}.dx^j$ it follows the corresponding form

$$\begin{aligned}\mathcal{L}_{\partial_k} dx^i &= k^i_{\ jk}.dx^j = f_j^{\ l}.f^i_{\ l,k}.dx^j, \\ k^i_{\ jk} &= f_j^{\ l}.f^i_{\ l,k}.\end{aligned}\quad (6)$$

On the other hand, from the commutation relations between S and the covariant differential operator ∇_ξ the connection between the partial derivatives of $f^i_{\ j}$ and the components of the contravariant and covariant connections Γ and P follows in the form

$$f^i_{\ l,k} = P^i_{\ mk}.f^m_{\ l} + \Gamma^m_{\ lk}.f^i_{\ m}.\quad (7)$$

After substituting the last expression in the expressions for $k^i_{\ j}(\xi)$ and for $k^i_{\ jk}$ the corresponding quantities are obtained in the forms

$$k^i_{\ j}(\xi) = f_j^{\ l}.\xi^k_{\ ,l}.f^i_{\ k} + (P^i_{\ jk} + f_j^{\ l}.\Gamma^m_{\ lk}.f^i_{\ m})\xi^k,\quad (8)$$

$$k^i_{\ j}(\partial_k) = k^i_{\ jk} = P^i_{\ jk} + f_j^{\ l}.\Gamma^m_{\ lk}.f^i_{\ m},$$

$$\mathcal{L}_\xi dx^i = [f_j^{\ l}.\xi^k_{\ ,l}.f^i_{\ k} + (P^i_{\ jk} + f_j^{\ l}.\Gamma^m_{\ lk}.f^i_{\ m}).\xi^k].dx^j,\quad (9)$$

$$\begin{aligned}\mathcal{L}_{\partial_k} dx^i &= k^i_{\ jk}.dx^j = \\ &= (P^i_{\ jk} + f_j^{\ l}.\Gamma^m_{\ lk}.f^i_{\ m}).dx^j.\end{aligned}\quad (10)$$

If we introduce the abbreviations

$$\xi^{\bar{i}}_{\ ,\underline{j}} = f^i_{\ k}.\xi^k_{\ ,l}.f_j^{\ l}, \quad \Gamma^{\bar{i}}_{\ \underline{jk}} = f_j^{\ l}.\Gamma^m_{\ lk}.f^i_{\ m},\quad (11)$$

then the Lie derivatives of covariant co-ordinate basic vector fields dx^i along the contravariant vector fields ξ and ∂_k can be written in the form

$$\begin{aligned}\mathcal{L}_\xi dx^i &= [\xi^{\bar{i}}_{\ ,\underline{j}} + (P^i_{\ jk} + \Gamma^{\bar{i}}_{\ \underline{jk}})\xi^k].dx^j, \\ \mathcal{L}_{\partial_k} dx^i &= (P^i_{\ jk} + \Gamma^{\bar{i}}_{\ \underline{jk}}).dx^j.\end{aligned}\quad (12)$$

2.3.2 Lie derivative of covariant non-co-ordinate basic vector fields

In analogous way as in the case of covariant co-ordinate basic vector fields the Lie derivatives of covariant non-co-ordinate basic vector fields can be obtained by means of the relations

$$\begin{aligned}\mathcal{L}_\xi[S(e^\alpha \otimes e_\beta)] &= S[\mathcal{L}_\xi(e^\alpha \otimes e_\beta)], \\ S(e^\alpha \otimes e_\beta) &= f^\alpha_{\ \beta}, \\ \mathcal{L}_\xi[S(e^\alpha \otimes e_\beta)] &= \xi^\gamma.e_\gamma f^\alpha_{\ \beta}, \\ S[\mathcal{L}_\xi(e^\alpha \otimes e_\beta)] &= S(\mathcal{L}_\xi e^\alpha \otimes e_\beta) + S(e^\alpha \otimes \mathcal{L}_\xi e_\beta), \\ \mathcal{L}_\xi e_\beta &= -(e_\beta \xi^\gamma + C_{\beta\delta}{}^\gamma \xi^\delta)e_\gamma = -\xi^\gamma_{\ //\beta}.e_\gamma, \\ \xi^\gamma_{\ //\beta} &= e_\beta \xi^\gamma + C_{\beta\delta}{}^\gamma \xi^\delta, \\ \mathcal{L}_\xi e^\alpha &= k^\alpha_{\ \gamma}(\xi).e^\gamma, \\ k^\alpha_{\ \gamma}(\xi).f^\gamma_{\ \beta} &= \xi^\gamma_{\ //\beta}.f^\alpha_{\ \gamma} + \xi^\gamma.e_\gamma f^\alpha_{\ \beta}, \\ f^\alpha_{\ \gamma}.f^\gamma_{\ \beta} &= g^\alpha_{\ \beta}, \quad f^\gamma_{\ \beta}.f_\gamma{}^\alpha = g^\alpha_{\ \beta}, \\ e_\gamma f^\alpha_{\ \delta} &= P^\alpha_{\ \sigma\gamma}.f^\sigma_{\ \delta} + \Gamma^\sigma_{\ \delta\gamma}.f^\alpha_{\ \sigma}, \\ k^\alpha_{\ \beta}(\xi) &= f^\alpha_{\ \gamma}.\xi^\gamma_{\ //\beta}.f^\beta_{\ \delta} + (P^\alpha_{\ \beta\gamma} + f_\beta{}^\delta.\Gamma^\sigma_{\ \delta\gamma}.f^\alpha_{\ \sigma})\xi^\gamma,\end{aligned}\quad (13)$$

$$\begin{aligned}\mathcal{L}_{e_\gamma} e^\alpha &= k^\alpha{}_\beta (e_\gamma) e^\beta = k^\alpha{}_{\beta\gamma} e^\beta, \\ k^\alpha{}_\beta (e_\gamma) &= k^\alpha{}_{\beta\gamma} = P^\alpha_{\beta\gamma} + f_\beta{}^\delta (\Gamma^\sigma_{\delta\gamma} + C_{\delta\gamma}{}^\sigma) f^\alpha{}_\sigma, \end{aligned} \quad (14)$$

in the form

$$\begin{aligned}\mathcal{L}_\xi e^\alpha &= [\xi^\alpha{}_{//\beta} + (P^\alpha_{\beta\gamma} + \Gamma^\alpha_{\beta\gamma}) \xi^\gamma] e^\beta = \\ &= [e_\beta \xi^\alpha + (P^\alpha_{\beta\gamma} + \Gamma^\alpha_{\beta\gamma} + C_{\beta\gamma}{}^\alpha) \xi^\gamma] e^\beta, \end{aligned} \quad (15)$$

$$\mathcal{L}_{e_\gamma} e^\alpha = (P^\alpha_{\beta\gamma} + \Gamma^\alpha_{\beta\gamma} + C_{\beta\gamma}{}^\alpha) e^\beta, \quad (16)$$

where

$$\begin{aligned}\xi^\alpha{}_{//\beta} &= f^\alpha{}_\gamma \xi^\gamma{}_{//\delta} f_\beta{}^\delta = f^\alpha{}_\gamma (e_\delta \xi^\gamma) f_\beta{}^\delta + f^\alpha{}_\gamma C_{\delta\sigma}{}^\gamma f_\beta{}^\delta \xi^\sigma = \\ &= e_\beta \xi^\alpha + C_{\beta\sigma}{}^\alpha \xi^\sigma, \\ e_\beta \xi^\alpha &= f^\alpha{}_\gamma (e_\delta \xi^\gamma) f_\beta{}^\delta, \quad C_{\beta\sigma}{}^\alpha = f^\alpha{}_\gamma C_{\delta\sigma}{}^\gamma f_\beta{}^\delta, \\ \Gamma^\alpha_{\beta\gamma} &= f_\beta{}^\delta \Gamma^\alpha_{\delta\gamma} f^\alpha{}_\sigma. \end{aligned} \quad (17)$$

2.4 Classification of linear transports with respect to the connections between contravariant and covariant affine connection

By means of the Lie derivatives of covariant basis vector fields a classification can be proposed for the connections between the components Γ^i_{jk} ($\Gamma^\alpha_{\beta\gamma}$) of the contravariant affine connection Γ and the components P^i_{jk} ($P^\alpha_{\beta\gamma}$) of the covariant affine connection P . On this basis, linear transports (induced by the covariant differential operator or by connections) and draggings along (induced by the Lie-differential operator) can be considered as connected each other through commutation relations of both the operators with the contraction operator.

Transport condition

$$\begin{aligned}P^\alpha_{\beta\gamma} + \Gamma^\alpha_{\beta\gamma} + C_{\beta\gamma}{}^\alpha &= \overline{F}^\alpha_{\beta\gamma}, \\ P^i_{jk} + \Gamma^i_{jk} &= \overline{F}^i_{jk}.\end{aligned}$$

$$\begin{aligned}P^\alpha_{\beta\gamma} + \Gamma^\alpha_{\beta\gamma} &= \overline{A}_\gamma g_\beta^\alpha, \\ P^i_{jk} + \Gamma^i_{jk} &= \overline{A}_k g_j^i.\end{aligned}$$

$$\begin{aligned}P^\alpha_{\beta\gamma} + \Gamma^\alpha_{\beta\gamma} &= 0, \\ P^i_{jk} + \Gamma^i_{jk} &= 0.\end{aligned}$$

Type of dragging along and transports

$$\begin{aligned}\mathcal{L}_{e_\gamma} e^\alpha &= \overline{F}^\alpha_{\beta\gamma} e^\beta, \\ \mathcal{L}_{\partial_k} dx^i &= \overline{F}^i_{jk} dx^j.\end{aligned}$$

Transport with arbitrary dragging along

$$\begin{aligned}\mathcal{L}_{e_\gamma} e^\alpha &= \overline{A}_\gamma e^\alpha + C_{\beta\gamma}{}^\alpha e^\beta, \\ \mathcal{L}_{\partial_k} dx^i &= \overline{A}_k dx^i.\end{aligned}$$

Transport with co-linear dragging along

$$\begin{aligned}\mathcal{L}_{e_\gamma} e^\alpha &= C_{\beta\gamma}{}^\alpha e^\beta, \\ \mathcal{L}_{\partial_k} dx^i &= 0.\end{aligned}$$

Transport with invariant dragging along

The classification of the connections on the basis of different transport conditions is analogous to the classification, proposed by Schouten and considered by (Schmutzer 1968).

2.5 Lie derivatives of covariant vector fields

The action of the Lie-differential operator on covariant vector and tensor fields is determined by its action on covariant basic vector fields and on the functions over M .

In co-ordinate basis the Lie derivative of covariant vector field p along a contravariant vector field ξ can be written in the forms

$$\begin{aligned}\mathcal{L}_\xi p &= \mathcal{L}_\xi(p_i dx^i) = (\mathcal{L}_\xi p_i) dx^i = \\ &= [p_{i,k} \xi^k + p_j \xi^j_{;i} + p_j (P_{ik}^j + \Gamma_{ik}^j) \xi^k] dx^i = \\ &= [p_{i;k} \xi^k + \xi^k_{;i} p_k + T_{ki}^j p_j \xi^k] dx^i ,\end{aligned}\tag{18}$$

where

$$\begin{aligned}\xi^j_{;i} &= f^j_{k} \xi^k_{;l} f_i^{l} , \quad T_{ki}^j = f^j_{l} T_{km}^l f_i^{m} , \\ T_{ki}^j &= \Gamma_{ik}^j - \Gamma_{ki}^j , \\ &(\text{in co-ordinate basis}).\end{aligned}\tag{19}$$

In non-co-ordinate basis the Lie derivative $\mathcal{L}_\xi p$ has the forms

$$\begin{aligned}\mathcal{L}_\xi p &= \mathcal{L}_\xi(p_\alpha e^\alpha) = (\mathcal{L}_\xi p_\alpha) e^\alpha = \\ &= \{(e_\gamma p_\alpha + P_{\alpha\gamma}^\beta p_\beta) \xi^\gamma + p_\beta [e_\alpha \xi^\beta + (\Gamma_{\alpha\gamma}^\beta + C_{\alpha\gamma}^\beta) \xi^\gamma]\} e^\alpha = \\ &= (p_{\alpha/\beta} \xi^\beta + \xi^\beta_{/\alpha} p_\beta + T_{\gamma\alpha}^\beta p_\beta \xi^\gamma) e^\alpha ,\end{aligned}\tag{20}$$

where

$$\begin{aligned}\xi^\beta_{/\alpha} &= f^\beta_{\delta} \xi^\delta_{/\gamma} f_\alpha^{\gamma} , \quad T_{\gamma\alpha}^\beta = f_\alpha^{\delta} T_{\gamma\delta}^\sigma f^\beta_{\sigma} , \\ T_{\beta\gamma}^\alpha &= \Gamma_{\gamma\beta}^\alpha - \Gamma_{\beta\gamma}^\alpha - C_{\beta\gamma}^\alpha , \\ &(\text{in non-co-ordinate basis}).\end{aligned}\tag{21}$$

The action of the Lie-differential operator on covariant tensor fields is determined by its action on basic tensor fields.

3 Kinematic characteristics connected with the notion relative velocity

3.1 Relative velocity

The notion *relative velocity* vector field (relative velocity) $_{rel}v$ can be defined as the orthogonal to a non-isotropic vector field u projection of the first covariant derivative (along the same non-isotropic vector field u) of (another) vector field ξ , i.e.

$$\begin{aligned}_{rel}v &= \bar{g}(h_u(\nabla_u \xi)) = g^{ij} h_{jk} \xi^k_{;l} u^l \cdot e_i = \\ &= g^{ij} f^m_{j} \cdot f^n_{k} h_{mn} \xi^k_{;l} u^l \cdot e_i , \\ e_i &= \partial_i \text{ (in co-ordinate basis),}\end{aligned}\tag{22}$$

where (the indices in co-ordinate and in non-co-ordinate basis are written in both cases as Latin indices instead as Latin and Greek indices)

$$h_u = g - \frac{1}{e}.g(u) \otimes g(u) , \quad (23)$$

$$h_u = h_{ij}.e^i.e^j, \bar{g} = g^{ij}.e_i.e_j, ,$$

$$\begin{aligned} \nabla_u \xi &= \xi^i{}_{;j} u^j . e_i \\ \xi^i{}_{;j} &= e_j \xi^i + \Gamma_{kj}^i \xi^k \\ \Gamma_{kj}^i &\neq \Gamma_{jk}^i , \end{aligned} \quad (24)$$

$$\begin{aligned} g &= g_{ij}.e^i.e^j, \\ g_{ij} &= g_{ji} , \\ e^i.e^j &= \frac{1}{2}(e^i \otimes e^j + e^j \otimes e^i) , \end{aligned} \quad (25)$$

$$\begin{aligned} e &= g(u, u) = g_{ij}^{-1} u^i u^j = u_i^{-1} u^i \neq 0 \\ g(u) &= g_{ik} u^k = u_i = g_{ik}.u^{\bar{k}} , u^{\bar{k}} = f^k{}_{\bar{l}} . u^{\bar{l}} , \\ e_i.e_j &= \frac{1}{2}(e_i \otimes e_j + e_j \otimes e_i) , \\ g_{ij} &= f^k{}_{\bar{i}} . f^{\bar{l}}{}_{\bar{j}} . g_{kl} , \\ g[\bar{g}(p)] &= p , p \in T^*(M) , \bar{g}[g(u)] = u , \\ g^{\bar{i}\bar{k}} g_{kj} &= g_j^{\bar{i}} , g_{ik}^{-1} g^{kj} = g_i^j , g^{\bar{i}\bar{j}} = f^i{}_{\bar{k}} . f^{\bar{j}}{}_{\bar{l}} . g^{kl} , \end{aligned} \quad (26)$$

$$\begin{aligned} h_u(\nabla_u \xi) &= h_{ij} \xi^j{}_{;k} u^k . e^i \\ h_{ij} &= g_{ij} - \frac{1}{e}.u_i u_j . \end{aligned} \quad (27)$$

In a co-ordinate basis

$$\begin{aligned} e_j \xi^i &= \xi^i{}_{;j} = \partial_j \xi^i = \partial \xi^i / \partial x^j , \\ e^j &= dx^j , \\ e_i &= \partial_i = \partial / \partial x^i , \\ u &= u^i . \partial_i , \end{aligned}$$

Every contravariant vector field ξ can be written by means of its projection along and orthogonal to u in two parts - one collinear to u and one - orthogonal to u , i.e.

$$\xi = \frac{l}{e}.u + h^u[g(\xi)] = \frac{l}{e}.u + \bar{g}[h_u(\xi)] , \quad (28)$$

where

$$\begin{aligned} l &= g(\xi, u) \\ h^u &= \bar{g} - \frac{1}{e}.u \otimes u \\ \xi &= \xi^i . \partial_i = \xi^k . e_k \\ h^u &= h^{ij} e_i . e_j , \end{aligned} \quad (29)$$

$$\begin{aligned} \bar{g}(h_u) \bar{g} &= h^u \\ h_u(\bar{g})(g) &= h_u \\ h^u(g)(\bar{g}) &= h^u \\ g(h^u)g &= h_u . \end{aligned} \quad (30)$$

Therefore, $\nabla_u \xi$ can be written in the form

$$\nabla_u \xi = \frac{\bar{l}}{e} \cdot u + \bar{g}[h_u(\nabla_u \xi)] = \frac{\bar{l}}{e} \cdot u + {}_{rel} v \quad (31)$$

$$\bar{l} = g(\nabla_u \xi, u)$$

and the connection between $\nabla_u \xi$ and ${}_{rel} v$ is obvious. Using the relation (Yano 1957) between the Lie derivative $\mathcal{L}_\xi u$ and the covariant derivative $\nabla_\xi u$

$$\begin{aligned} \mathcal{L}_\xi u &= \nabla_\xi u - \nabla_u \xi - T(\xi, u) \\ T(\xi, u) &= T_{ij}^k \xi^i u^j \cdot e_k, \end{aligned} \quad (32)$$

$$T_{ij}^k = -T_{ji}^k = \Gamma_{ji}^k - \Gamma_{ij}^k - C_{ij}^k$$

(in a non-co-ordinate basis $\{e_k\}$),

$$[e_i, e_j] = \mathcal{L}_{e_i} e_j = C_{ij}^k \cdot e_k,$$

$$T_{ij}^k = \Gamma_{ji}^k - \Gamma_{ij}^k$$

(in a co-ordinate basis $\{\partial_k\}$),

one can write $\nabla_u \xi$ in the form

$$\nabla_u \xi = (k)g(\xi) - \mathcal{L}_\xi u = k[g(\xi)] - \mathcal{L}_\xi u, \quad (33)$$

or taking into account the above expression for ξ - in the form

$$\nabla_u \xi = k[h_u(\xi)] + \frac{l}{e} \cdot a - \mathcal{L}_\xi u,$$

where

$$\begin{aligned} k[g(\xi)] &= \nabla_\xi u - T(\xi, u) \\ k &= (u^i \cdot l - T_{lk}^i u^k) g^{lj} \cdot e_i \otimes e_j, \end{aligned} \quad (34)$$

$$\begin{aligned} k[g(u)] &= k(g)u = k^{ij} g_{jk} u^k \cdot e_i \\ &= a = \nabla_u u = u^i \cdot j u^j \cdot e_i. \end{aligned} \quad (35)$$

For $h_u(\nabla_u \xi)$ it follows

$$h_u(\nabla_u \xi) = h_u\left(\frac{l}{e} \cdot a - \mathcal{L}_\xi u\right) + h_u(k)h_u(\xi), \quad (36)$$

where

$$\begin{aligned} h_u(k)h_u(\xi) &= h_{ik} k^{kl} h_{lj} \xi^j \cdot e^i, \\ h_u(u) &= 0, \\ u(h_u) &= 0, \\ h_u(k)h_u(u) &= 0, \\ (u)h_u(k)h_u &= 0. \end{aligned}$$

If we introduce the abbreviation

$$d = h_u(k)h_u = h_{ik}k^{kl}h_{lj}.e^i \otimes e^j = d_{ij}.e^i \otimes e^j , \quad (37)$$

the expression for ${}_{rel}v$ can take the form

$$\begin{aligned} {}_{rel}v &= \bar{g}[h_u(\nabla_u \xi)] = \bar{g}(h_u)\left(\frac{l}{e}.a - \mathcal{L}_\xi u\right) + \bar{g}[d(\xi)] = \\ &= [g^{ik}h_{kl}\left(\frac{l}{e}.a^l - \mathcal{L}_\xi u^l\right) + g^{ik}d_{kl}\xi^l].e_i = {}_{rel}v^i.e_i , \end{aligned} \quad (38)$$

or

$$g({}_{rel}v) = h_u(\nabla_u \xi) = h_u\left(\frac{l}{e}.a - \mathcal{L}_\xi u\right) + d(\xi) . \quad (39)$$

For the special case when the vector field ξ is orthogonal to u , i.e. $\xi = \bar{g}[h_u(\xi)]$, and the Lie derivative of u along ξ is zero, i.e. $\mathcal{L}_\xi u = 0$, then the relative velocity can be written in the form

$$g({}_{rel}v) = d(\xi) \quad (40)$$

or in the form

$${}_{rel}v = \bar{g}[d(\xi)].$$

3.2 Deformation velocity, shear velocity, rotation velocity and expansion velocity

The covariant tensor field d is a generalization for (\overline{L}_n, g) -spaces of the well known *deformation velocity* tensor for V_n -spaces (Stephani 1977), (Kramer, Stephani, MacCallum, Herlt 1980). It is usually represented by means of its three parts: the trace-free symmetric part, called *shear velocity* tensor (shear), the anti symmetric part, called *rotation velocity* tensor (rotation) and the trace part, in which the trace is called *expansion velocity* (expansion) invariant.

After some more complicated as for V_n -spaces calculations the deformation velocity tensor d can be given in the form

$$\begin{aligned} d &= h_u(k)h_u = h_u(k_s)h_u + h_u(k_a)h_u = \\ &= \sigma + \omega + \frac{1}{n-1}.\theta.h_u , \end{aligned} \quad (41)$$

where

σ is the *shear velocity* tensor (shear) ,

$$\begin{aligned} \sigma &= {}_sE - {}_sP = E - P - \frac{1}{n-1}.\bar{g}[E - P].h_u = \sigma_{ij}.e^i.e^j = \\ &= E - P - \frac{1}{n-1}.\theta_o - \theta_1.h_u , \end{aligned} \quad (42)$$

$$\begin{aligned} {}_sE &= E - \frac{1}{n-1}.\bar{g}[E].h_u \\ \bar{g}[E] &= g^{ij}.E_{ij} = g^{\bar{i}\bar{j}}.E_{\bar{i}\bar{j}} = \theta_o , \end{aligned} \quad (43)$$

$$\begin{aligned}
E &= h_u(\epsilon)h_u \\
k_s &= \epsilon - m \\
\epsilon &= \frac{1}{2}(u^i{}_{;l}g^{lj} + u^j{}_{;l}g^{li}).e_i.e_j ,
\end{aligned} \tag{44}$$

$$m = \frac{1}{2}(T_{lk}^i u^k g^{lj} + T_{lk}^j u^k g^{li})e_i.e_j . \tag{45}$$

${}_sE$ is the *torsion-free shear velocity* tensor, ${}_sP$ is the *shear velocity* tensor induced by the torsion,

$$\begin{aligned}
{}_sP &= P - \frac{1}{n-1}.\overline{g}[P].h_u \\
\overline{g}[P] &= g^{kl}P_{kl} = g^{\overline{k}\overline{l}}.P_{kl} = \theta_1,
\end{aligned} \tag{46}$$

$$\begin{aligned}
P &= h_u(m)h_u \\
\theta_1 &= T_{kl}^k u^l \\
\theta_o &= u^n{}_{;n} - \frac{1}{2e}(e_{,k}u^k - g_{kl;m}u^m u^{\overline{k}}u^{\overline{l}}) ,
\end{aligned} \tag{47}$$

$$\begin{aligned}
e_{,k} &= e_k e \\
\theta &= \theta_o - \theta_1 ,
\end{aligned} \tag{48}$$

θ is the *expansion velocity*, θ_o is the *torsion-free expansion velocity*, θ_1 is the *expansion velocity induced by the torsion*,
 ω is the *rotation velocity* tensor (rotation velocity),

$$\omega = h_u(k_a)h_u = h_u(s)h_u - h_u(q)h_u = S - Q , \tag{49}$$

$$\begin{aligned}
s &= \frac{1}{2}(u^k{}_{;m}g^{ml} - u^l{}_{;m}g^{mk}).e_k \wedge e_l \\
e_k \wedge e_l &= \frac{1}{2}(e_k \otimes e_l - e_l \otimes e_k) ,
\end{aligned} \tag{50}$$

$$\begin{aligned}
q &= \frac{1}{2}(T_{mn}^k g^{ml} - T_{mn}^l g^{mk})u^n.e_k \wedge e_l \\
S &= h_u(s)h_u , Q = h_u(q)h_u ,
\end{aligned} \tag{51}$$

S is the *torsion-free rotation velocity* tensor, Q is the *rotation velocity* tensor induced by the torsion.

By means of the expressions for σ , ω and θ the deformation velocity tensor can be written in two parts

$$\begin{aligned}
d &= d_o + d_1 \\
d_o &= {}_sE + S + \frac{1}{n-1}.\theta_o.h_u \\
d_1 &= {}_sP + Q + \frac{1}{n-1}.\theta_1.h_u ,
\end{aligned} \tag{52}$$

where d_o is the *torsion-free deformation velocity* tensor and d_1 is the *deformation velocity tensor induced by the torsion*. For the case of V_n -spaces $d_1 = 0$ (${}_sP = 0$, $Q = 0$, $\theta_1 = 0$).

The shear velocity tensor σ and the expansion velocity θ can be written also in the form

$$\sigma = \frac{1}{2} \{ h_u (\nabla_u \bar{g} - \mathcal{L}_u \bar{g}) h_u - \frac{1}{n-1} (h_u [\nabla_u \bar{g} - \mathcal{L}_u \bar{g}]) \cdot h_u \} = \quad (53)$$

$$= \frac{1}{2} \{ h_{i\bar{k}} (g^{kl}{}_{;m} u^m - \mathcal{L}_u g^{kl}) h_{\bar{l}j} - \frac{1}{n-1} \cdot h_{\bar{k}l} (g^{kl}{}_{;m} u^m - \mathcal{L}_u g^{kl}) \cdot h_{ij} \} \cdot e^i \cdot e^j . \quad (54)$$

$$\begin{aligned} \theta &= \frac{1}{2} \cdot h_u [\nabla_u \bar{g} - \mathcal{L}_u \bar{g}] = \frac{1}{2} [\nabla_{\bar{g}} u + T(u, \bar{g})] = \\ &= \frac{1}{2} h_{i\bar{j}} (g^{ij}{}_{;k} u^k - \mathcal{L}_u g^{ij}) . \end{aligned}$$

The main result of the above considerations can be summarized in the following proposition:

Proposition 2. The covariant vector field $g_{(rel)v} = h_u(\nabla_u \xi)$ can be written in the forms:

$$\begin{aligned} h_u(\nabla_u \xi) &= h_u \left(\frac{l}{e} \cdot a - \mathcal{L}_\xi u \right) + d(\xi) = \\ &= h_u \left(\frac{l}{e} \cdot a - \mathcal{L}_\xi u \right) + \sigma(\xi) + \omega(\xi) + \frac{1}{n-1} \cdot \theta \cdot h_u(\xi) . \end{aligned}$$

The physical interpretation of the velocity tensors d, σ, ω and of the invariant θ for the case of V_4 -spaces (Synge 1960), (Ehlers 1961), (Kramer, Stephani, MacCallum, Herlt 1980) can be extended also for (\bar{L}_4, g) -spaces. In this case the torsion play an equivalent role as the covariant derivative in the velocity tensors. The individual designation, connected with the physical interpretation of these kinematic characteristics, is given in the Appendix A - Table 1. It is easy to see that the existence of some kinematic characteristics (${}_sP, Q, \theta_1$) depends on the existence of the torsion tensor field. They vanish if it is equal to zero (e.g. in V_n -spaces).

4 Kinematic characteristics connected with the notion relative acceleration

4.1 Relative acceleration

The notion *relative acceleration* vector field (relative acceleration) ${}_{rel}a$ can be defined (in analogous way as ${}_{rel}v$) as the orthogonal to a non-isotropic vector field u ($g(u, u) = e \neq 0$) projection of the second covariant derivative (along the same non-isotropic vector field u) of (another) vector field ξ , i.e.

$${}_{rel}a = \bar{g}(h_u(\nabla_u \nabla_u \xi)) = g^{ij} h_{j\bar{k}} (\xi^k{}_{;l} u^l)_{;m} u^m e_i . \quad (55)$$

$\nabla_u \nabla_u \xi = (\xi^i{}_{;l} u^l)_{;m} u^m e_i$ is the second covariant derivative of a vector field ξ along the vector field u . It is an essential part of all types of deviation equations in V_n - and (L_n, g) -spaces (Manoff 1979, 1984), (Iliev, Manoff 1983).

If we take into account the expression for $\nabla_u \xi$

$$\nabla_u \xi = k[g(\xi)] - \mathcal{L}_\xi u,$$

and differentiate covariant along u , then we obtain

$$\nabla_u \nabla_u \xi = \{\nabla_u[(k)g]\}(\xi) + (k)(g)(\nabla_u \xi) - \nabla_u(\mathcal{L}_\xi u)$$

By means of the relations

$$k(g)\bar{g} = k ,$$

$$\nabla_u[k(g)] = (\nabla_u k)(g) + k(\nabla_u g) ,$$

$$\{\nabla_u[k(g)]\}\bar{g} = \nabla_u k + k(\nabla_u g)\bar{g} ,$$

$\nabla_u \nabla_u \xi$ can be written in the form

$$\nabla_u \nabla_u \xi = \frac{l}{e}.H(u) + B(h_u)\xi - k(g)\mathcal{L}_\xi u - \nabla_u(\mathcal{L}_\xi u) \quad (56)$$

$$(\text{compare with } \nabla_u \xi = \frac{l}{e}.a + k(h_u)\xi - \mathcal{L}_\xi u),$$

where

$$H = B(g) = (\nabla_u k)(g) + k(\nabla_u g) + k(g)k(g) ,$$

$$B = \nabla_u k + k(g)k + k(\nabla_u g)\bar{g} = \nabla_u k + k(g)k - k(g)(\nabla_u \bar{g}) .$$

The orthogonal to u covariant projection of $\nabla_u \nabla_u \xi$ will have therefore the form

$$h_u(\nabla_u \nabla_u \xi) = h_u[\frac{l}{e}.H(u) - k(g)\mathcal{L}_\xi u - \nabla_u \mathcal{L}_\xi u] + [h_u(B)h_u](\xi) . \quad (57)$$

In the special case, when $g(u, \xi) = l = 0$ and $\mathcal{L}_\xi u = 0$, the above expression has the simple form

$$h_u(\nabla_u \nabla_u \xi) = [h_u(B)h_u](\xi) = A(\xi) , \quad (58)$$

$$(\text{compare with } h_u(\nabla_u \xi) = [h_u(k)h_u](\xi) = d(\xi)).$$

The explicit form of $H(u)$ follows from the explicit form of H and its action on the vector field u

$$H(u) = (\nabla_u k)[g(u)] + k(\nabla_u g)(u) + k(g)(a) = \nabla_u[k(g)(u)] = \nabla_u a . \quad (59)$$

Now $h_u[\nabla_u \nabla_u \xi]$ can be written in the form

$$h_u(\nabla_u \nabla_u \xi) = h_u[\frac{l}{e}.\nabla_u a - k(g)(\mathcal{L}_\xi u) - \nabla_u(\mathcal{L}_\xi u)] + A(\xi) \quad (60)$$

$$(\text{compare } h_u(\nabla_u \xi) = h_u(\frac{l}{e}.a - \mathcal{L}_\xi u) + d(\xi)).$$

The explicit form of $A = h_u(B)h_u$ can be found in analogous way as the explicit form for $d = h_u(k)h_u$ in the expression for $rel v$.

4.2 Deformation acceleration, shear acceleration, rotation acceleration and expansion acceleration

The covariant tensor A , named *deformation acceleration* tensor can be represented as a sum, containing three terms: a trace-free symmetric term, an anti symmetric term and a trace term

$$A = {}_s D + W + \frac{1}{n-1} \cdot U \cdot h_u \quad (61)$$

where

$$D = h_u({}_s B)h_u \quad (62)$$

$$W = h_u({}_a B)h_u \quad (63)$$

$$U = \bar{g}[_s A] = \bar{g}[D] \quad (64)$$

$${}_s B = \frac{1}{2}(B^{ij} + B^{ji})e_i \cdot e_j, \quad {}_a B = \frac{1}{2}(B^{ij} - B^{ji})e_i \wedge e_j, \quad (65)$$

$${}_s A = \frac{1}{2}(A_{ij} + A_{ji})e^i \cdot e^j, \quad (66)$$

$${}_s D = D - \frac{1}{n-1} \cdot \bar{g}[D] \cdot h_u = D - \frac{1}{n-1} \cdot U \cdot h_u. \quad (67)$$

${}_s D$ is the *shear acceleration* tensor (shear acceleration), W is the *rotation acceleration* tensor (rotation acceleration) and U is the *expansion acceleration* invariant (expansion acceleration). Furthermore, every one of these quantities can be divided into three parts: torsion- and curvature-free acceleration, acceleration induced by torsion and acceleration induced by curvature.

Let us now consider the representation of every acceleration quantity in its essential parts, connected with its physical interpretation.

The deformation acceleration tensor A can be written in the following forms

$$A = {}_s D + W + \frac{1}{n-1} \cdot U \cdot h_u = A_0 + G = {}_F A_0 - {}_T A_0 + G, \quad (68)$$

$$A = {}_s D_0 + W_0 + \frac{1}{n-1} \cdot U_0 \cdot h_u + {}_s M + N + \frac{1}{n-1} \cdot I \cdot h_u, \quad (69)$$

$$\begin{aligned} A = & {}_s F D_0 + {}_F W_0 + \frac{1}{n-1} \cdot {}_F U_0 \cdot h_u - \\ & - ({}_s T D_0 + {}_T W_0 + \frac{1}{n-1} \cdot {}_T U_0 \cdot h_u) + \\ & + {}_s M + N + \frac{1}{n-1} \cdot I \cdot h_u, \end{aligned} \quad (70)$$

where

$$A_0 = {}_F A_0 - {}_T A_0 = {}_s D_0 + W_0 + \frac{1}{n-1} \cdot U_0 \cdot h_u, \quad (71)$$

$${}_F A_0 = {}_s F D_0 + {}_F W_0 + \frac{1}{n-1} \cdot {}_F U_0 \cdot h_u, \quad (72)$$

$${}_F A_0(\xi) = h_u(\nabla_{\xi_\perp} a), \quad \xi_\perp = \bar{g}[h_u(\xi)], \quad (73)$$

$${}_T A_0 = {}_s T D_0 + {}_T W_0 + \frac{1}{n-1} \cdot {}_T U_0 \cdot h_u , \quad (74)$$

$$G = {}_s M + N + \frac{1}{n-1} \cdot I \cdot h_u = h_u(K)h_u , \quad (75)$$

$$h_u([R(u, \xi)]u) = h_u(K)h_u(\xi) \text{ for } \forall \xi \in T(M) , \quad (76)$$

$$[R(u, \xi)]u = \nabla_u \nabla_\xi u - \nabla_\xi \nabla_u u - \nabla_{\mathcal{L}_u \xi} u , \quad (77)$$

$$K = K^{kl} e_k \otimes e_l , K^{kl} = R^k{}_{mnr} g^{rl} u^m u^n , \quad (78)$$

$R^k{}_{mnr}$ are the components of the contravariant Riemannian curvature tensor,

$$K_a = K_a^{kl} \cdot e_k \wedge e_l , K_a^{kl} = \frac{1}{2} (K^{kl} - K^{lk}) , K_s = K_s^{kl} e_k \cdot e_l , K_s^{kl} = \frac{1}{2} (K^{kl} + K^{lk}) , \quad (79)$$

$${}_s D = {}_s D_0 + {}_s M , W = W_0 + N = {}_F W_0 - {}_T W_0 + N , \quad (80)$$

$$U = U_0 + I = {}_F U_0 - {}_T U_0 + I , \quad (81)$$

$${}_s M = M - \frac{1}{n-1} \cdot I \cdot h_u , M = h_u(K_s)h_u , I = \bar{g}[M] = g^{ij} M_{ij} , \quad (82)$$

$$N = h_u(K_a)h_u , \quad (83)$$

$${}_s D_0 = {}_s F D_0 - {}_s T D_0 = {}_F D_0 - \frac{1}{n-1} \cdot {}_F U_0 \cdot h_u - ({}_T D_0 - \frac{1}{n-1} \cdot {}_T U_0 \cdot h_u) , \quad (84)$$

$${}_s D_0 = {}_s F D_0 - {}_T D_0 - \frac{1}{n-1} ({}_F U_0 - {}_T U_0) h_u , \quad (85)$$

$${}_s D_0 = D_0 - \frac{1}{n-1} \cdot U_0 \cdot h_u , \quad (86)$$

$${}_s F D_0 = {}_F D_0 - \frac{1}{n-1} \cdot {}_F U_0 \cdot h_u , {}_F D_0 = h_u(b_s)h_u , \quad (87)$$

$$b = b_s + b_a , b = b^{kl} e_k \otimes e_l , b^{kl} = a^k{}_{;n} g^{nl} , \quad (88)$$

$$a^k = u^k{}_{;m} u^m , b_s = b_s^{kl} e_k \cdot e_l , b_s^{kl} = \frac{1}{2} (b^{kl} + b^{lk}) , \quad (89)$$

$$b_a = b_a^{kl} e_k \wedge e_l , b_a^{kl} = \frac{1}{2} (b^{kl} - b^{lk}) , \quad (90)$$

$${}_F U_0 = \bar{g}[{}_F D_0] = g[b] - \frac{1}{c} \cdot g(u, \nabla_u a) , g[b] = g_{kl} b^{kl} , \quad (91)$$

$${}_s T D_0 = {}_T D_0 - \frac{1}{n-1} \cdot {}_T U_0 \cdot h_u = {}_s F D_0 - {}_s D_0 , {}_T D_0 = {}_F D_0 - D_0 , \quad (92)$$

$$U_0 = \bar{g}[D_0] = {}_F U_0 - {}_T U_0 , {}_T U_0 = \bar{g}[{}_T D_0] , \quad (93)$$

$${}_F W_0 = h_u(b_a)h_u , {}_T W_0 = {}_F W_0 - W_0 . \quad (94)$$

Under the conditions $\mathcal{L}_\xi u = 0$, $\xi = \xi_\perp = \overline{g}(h_u(\xi))$, ($l = 0$), the expression for $h_u(\nabla_u \nabla_u \xi)$ can be written in the forms

$$h_u(\nabla_u \nabla_u \xi_\perp) = A(\xi_\perp) = A_0(\xi_\perp) + G(\xi_\perp) , \quad (95)$$

$$h_u(\nabla_u \nabla_u \xi_\perp) = {}_F A_0(\xi_\perp) - {}_T A_0(\xi_\perp) + G(\xi_\perp) , \quad (96)$$

$$\begin{aligned} h_u(\nabla_u \nabla_u \xi_\perp) = & ({}_s F D_0 + {}_F W_0 + \frac{1}{n-1} \cdot {}_F U_0 \cdot g)(\xi_\perp) - \\ & - ({}_s T D_0 + {}_T W_0 + \frac{1}{n-1} \cdot {}_T U_0 \cdot g)(\xi_\perp) + ({}_s M + N + \frac{1}{n-1} \cdot {}_I g)(\xi_\perp) , \end{aligned} \quad (97)$$

which enable one to find a physical interpretation of the quantities ${}_s D, W, U$ and of the contained in their structure quantities ${}_s F D_0, {}_F W_0, {}_F U_0, {}_s T D_0, {}_T W_0, {}_T U_0, {}_s M, N, I$. The individual designation, connected with their physical interpretation, is given in the Appendix A - Table 1. The expressions of these quantities in terms of the kinematic characteristics of the relative velocity are given in the Appendix A.

After the above consideration the following proposition can be formulated:

Proposition 3. $g(\text{rel}a) = h_u(\nabla_u \nabla_u \xi)$ can be written in the form

$$g(\text{rel}a) = h_u \left[\frac{l}{e} \cdot \nabla_u a - \nabla_{\mathcal{L}_\xi u} u - \nabla_u (\mathcal{L}_\xi u) + T(\mathcal{L}_\xi u, u) \right] + A(\xi) ,$$

where

$$A(\xi) = {}_s D(\xi) + W(\xi) + \frac{1}{n-1} \cdot U \cdot h_u(\xi) . \quad (98)$$

For the case of affine symmetric connection ($T(w, v) = 0$ for $\forall w, v \in T(M)$, $T_{ij}^k = 0, \Gamma_{ij}^k = \Gamma_{ji}^k$) and Riemannian metric ($\nabla_v g = 0$ for $\forall v \in T(M)$, $g_{ij;k} = 0$) kinematic characteristics are obtained in V_n -spaces, connected with the notion relative velocity (Manoff 1992) and relative acceleration (Manoff 1985). For the case of affine non-symmetric connection ($T(w, v) \neq 0$ for $\forall w, v \in T(M)$, $\Gamma_{jk}^i \neq \Gamma_{kj}^i$) and Riemannian metric kinematic characteristics are obtained in U_n -spaces (Manoff 1985).

5 Classification of auto-parallel vector fields on the basis of the kinematic characteristics connected with the relative velocity and relative acceleration

The classification of (pseudo)Riemannian spaces V_n , admitting the existence of auto-parallel (in the case of V_n -spaces they are geodesic) vector fields ($\nabla_u u = a = 0$) with given kinematic characteristics, connected with the notion relative velocity, can be extended to a classification of differentiable manifolds with contravariant and covariant affine connection and metric, admitting auto parallel vector fields with certain kinematic characteristics, connected with the relative

velocity and the relative acceleration. In this way the following two schemes for the existence of special type 1. and 2. of vector fields can be proposed (s. Appendix B. - Table 2.). Different types of combinations between the single conditions of the two schemes can also be taken under consideration.

5.1 Special geodesic vector fields with vanishing kinematic characteristics, induced by the curvature, in (pseudo) Riemannian spaces

On the basis of the classification 2. the following propositions in the case of V_n -spaces can be proved:

Proposition 4. Non-isotropic geodesic vector fields in V_n -spaces are geodesic vector fields with curvature rotation acceleration tensor N equal to zero, i.e. $N = 0$.

Proof:

$$N = h_u(K_a)h_u = h_{ik}K_a^{kl}h_{lj}e^i \wedge e^j, \quad (99)$$

$$K_a^{kl} = \frac{1}{2}(K^{kl} - K^{lk}) = \frac{1}{2}(R_{mnr}^k g^{rl} - R_{mnr}^l g^{rk})u^m u^n,$$

For the case of V_n -space, where

$$R_{kmnr} = R_{nrkm}, R_{kmnr} = g_{kl}R^l{}_{mnr}, \quad (100)$$

the conditions

$$R^k{}_{mnr}g^{rl} = R^l{}_{nmr}g^{rk}, R^k{}_{mn}{}^l = R^l{}_{nm}{}^k, \quad (101)$$

follow and therefore

$$K_a^{kl} = \frac{1}{2}(R^k{}_{mnr}g^{rl} - R^l{}_{mnr}g^{rk})u^m u^n = 0, \quad (102)$$

$$K_a = 0, N = 0.$$

Proposition 5. Non-isotropic geodesic vector fields in V_n -spaces with equal to zero Ricci tensor ($R_{ik} = R^l{}_{ikl} = g_m^l R^m{}_{ikl} = 0$) are geodesic vector fields with curvature rotation acceleration N and curvature expansion acceleration I , both equal to zero, i.e. $N = 0, I = 0$.

Proof: 1. From the proposition 4. it follows that $K_a = 0$ and $N = 0$.

$$2. I = g[K] = g_{ij}K^{ij} = g_{ij}R^i{}_{mnr}g^{rj}u^m u^n = g_i^r R^i{}_{mnr}u^m u^n = R_{mn}u^m u^n = 0. \quad (103)$$

Proposition 6. Non-isotropic geodesic vector fields in V_n -spaces with constant curvature

$$[R(\xi, \eta)]v = \frac{1}{n(n-1)}R_0[g(v, \xi)\eta - g(v, \eta)\xi], \forall \xi, \eta, v \in T(M), \quad (104)$$

(in index form

$$R^i{}_{jkl} = \frac{R_0}{n(n-1)}(g_l^i g_{jk} - g_k^i g_{jl}), R_0 = const.) \quad (105)$$

are geodesic vector fields with curvature shear acceleration and curvature rotation acceleration, both equal to zero, i.e. ${}_sM = 0$, $N = 0$.

Proof: 1. From the proposition 4. it follows that $N = 0$.

$$2. {}_sM = M - \frac{1}{n-1} \cdot I \cdot h_u, \quad M = h_u(K_s)h_u = g(K_s)g,$$

$$M = g(K_s)g = g_{ik}K^{kl}g_{lj}e^i \cdot e^j = M_{ij}e^i \cdot e^j,$$

$$M_{ij} = g_{ik}R^k{}_{mnr}g^{rl}g_{lj}u^m u^n = R_{imnj}u^m u^n = \frac{R_0}{n(n-1)} \cdot e \cdot h_{ij}, \quad (106)$$

$$M = \frac{R_0 \cdot e}{n(n-1)} \cdot h_u, \quad e = g(u, u) = g_{ij}u^i u^j, \quad (107)$$

$$I = g[K] = \bar{g}[M] = g^{ij}M_{ij} = \frac{1}{n} \cdot R_0 \cdot e, \quad g^{ij}h_{ij} = n-1, \quad (108)$$

$${}_sM = M - \frac{1}{n-1} \cdot I \cdot h_u = 0. \quad (109)$$

The projections of the curvature tensor of the type $G = h_u(K)h_u$ (or $R^i{}_{jkl}u^j u^k$) along the non-isotropic vector field u acquire a natural physical meaning as quantities, connected with the kinematic characteristics curvature shear acceleration ${}_sM$, curvature rotation acceleration N and curvature expansion acceleration I .

The projection of the Ricci tensor ($g[K]$, or $R_{ik}u^i u^k$) and the Raychaudhuri identity for vector fields represent an expression of the curvature expansion acceleration, given in terms of the kinematic characteristics of the relative velocity

$$\begin{aligned} I = \bar{g}[M] &= R_{ij}u^i u^j = \\ &= -a^j{}_{;j} + g^{ij} \cdot {}_sE_{ik} \cdot g^{kl} \cdot \sigma_{lj} + g^{ij} S_{ik} g^{kl} \omega_{lj} + \theta_0 + \frac{1}{n-1} \cdot \theta_0 \cdot \theta + \\ &\quad + \frac{1}{e} [a^k (e_{,k} - u^{\bar{n}} T_{km}^n u^m - g_{mn;k} u^{\bar{m}} u^{\bar{n}} - g_{k\bar{m};l} u^l u^{\bar{m}}) + \\ &\quad + \frac{1}{2} (u^k e_{,k})_{;l} u^l - \frac{1}{2} (g_{mn;k} u^k)_{;l} u^l u^{\bar{m}} u^{\bar{n}}] - \\ &\quad \frac{1}{e^2} [\frac{3}{4} (e_{,k} u^k)^2 - (e_{,k} u^k) g_{mn;l} u^l u^{\bar{m}} u^{\bar{n}} + \frac{1}{4} (g_{mn;l} u^l u^{\bar{m}} u^{\bar{n}})^2], \\ &\quad \theta \cdot = \theta_{,k} u^k \end{aligned} \quad (110)$$

In the case of V_n -spaces the kinematic characteristics, connected with the relative velocity and the relative acceleration have the forms:

a) kinematic characteristics, connected with the relative velocity

$$\begin{aligned} d &= d_0 & d_1 &= 0 & k &= k_o \\ \sigma &= {}_sE & {}_sP &= 0 & m &= 0 \\ \omega &= S & Q &= 0 & q &= 0 \\ \theta &= \theta_o & \theta_1 &= 0 & \nabla_u u &= a \neq 0, \quad a = 0 \end{aligned}$$

b) kinematic characteristics, connected with the relative acceleration ($\nabla_u u = a \neq 0$)

$$\begin{aligned} A &= {}_F A_0 + G & {}_T A_0 &= 0 & N &= 0 \\ G &= {}_sM + \frac{1}{n-1} \cdot I \cdot h_u & {}_sT D_0 &= 0 \\ W &= {}_F W_0 & {}_T W_0 &= 0 \\ U &= {}_F U_0 + I & {}_T U_0 &= 0 & \nabla_u u &= a \neq 0 \end{aligned}$$

c) kinematic characteristics, connected with the relative acceleration ($\nabla_u u = a = 0$)

$$\begin{aligned} A &= G & {}_T A_0 &= 0 & N &= 0 \\ G &= {}_s M + \frac{1}{n-1} \cdot I \cdot h_u & {}_{sT} D_0 &= 0 \\ W &= 0 & {}_T W_0 &= 0 \\ U &= I & {}_T U_0 &= 0 & \nabla_u u = a = 0 \end{aligned}$$

On the basis of the different kinematic characteristics dynamic systems can be classified and considered in V_n -spaces.

5.2 Special vector fields over manifolds with contravariant and covariant affine connection and metric with vanishing kinematic characteristics induced by the curvature

The explicit forms of the quantities G , M , N and I , connected with accelerations induced by curvature can be used for finding conditions for existence of special types of contravariant vector fields with vanishing characteristics induced by the curvature. G , M , N and I can be expressed in the following forms:

$$\begin{aligned} G &= h_u(K)h_u = g(K)g - \frac{1}{e} \cdot g(u) \otimes [g(u)](K)g, \\ K[g(u)] &= 0, \end{aligned} \quad (111)$$

$$\begin{aligned} M &= h_u(K_s)h_u = g(K_s)g - \frac{1}{2e} \{g(u) \otimes [g(u)](K)g + [g(u)](K)g \otimes g(u)\} = \\ &= M_{ij} \cdot dx^i \cdot dx^j = M_{\alpha\beta} \cdot e^\alpha \cdot e^\beta, \quad M_{ij} = M_{ji}, \\ M_{ij} &= \frac{1}{2} [g_{i\bar{k}} \cdot g_{\bar{l}j} + g_{j\bar{k}} \cdot g_{\bar{l}i} - \frac{1}{e} (u_i \cdot g_{\bar{l}j} + u_j \cdot g_{\bar{l}i}) u_{\bar{k}}] R^k{}_{mnq} u^m u^n \cdot g^{ql}, \end{aligned} \quad (112)$$

$$I = \bar{g}[M] = g[K_s] = g[K] = R_{\rho\sigma} \cdot u^\rho u^\sigma = R_{kl} \cdot u^k u^l, \quad (113)$$

$$\begin{aligned} N &= h_u(K_a)h_u = g(K_a)g - \frac{1}{2e} \{g(u) \otimes [g(u)](K)g - [g(u)](K)g \otimes g(u)\} = \\ &= N_{ij} \cdot dx^i \wedge dx^j = N_{\alpha\beta} \cdot e^\alpha \wedge e^\beta, \quad N_{ij} = -N_{ji}, \\ N_{ij} &= \frac{1}{2} [g_{i\bar{k}} \cdot g_{\bar{l}j} - g_{j\bar{k}} \cdot g_{\bar{l}i} - \frac{1}{e} (u_i \cdot g_{\bar{l}j} - u_j \cdot g_{\bar{l}i}) u_{\bar{k}}] R^k{}_{mnq} \cdot u^m u^n \cdot g^{ql}. \end{aligned} \quad (114)$$

By means of the above expressions conditions can be found under which some of the quantities M , N , I vanish.

5.2.1 Contravariant vector fields without rotation acceleration, induced by the curvature ($N = 0$)

If the rotation acceleration N , induced by the curvature vanishes, i.e. if $N = 0$, then the following proposition can be proved:

Proposition 7. The necessary and sufficient condition for the existence of a contravariant vector field u ($g(u, u) = e \neq 0$) without rotation acceleration, induced by the curvature (i.e. with $N = 0$) is the condition

$$K_a = \frac{1}{2e} \{u \otimes [g(u)](K) - [g(u)](K) \otimes u\}. \quad (115)$$

Proof: 1. Sufficiency: From the above expression it follows

$$\begin{aligned} N &= h_u(K_a)h_u = \\ &= g(K_a)g - \frac{1}{2e}\{g(u) \otimes [g(u)](K)g - [g(u)](K)g \otimes g(u)\} = 0 , \\ &\quad g([g(u)](K)) = [g(u)](K)g . \end{aligned}$$

2. Necessity: If $N = h_u(K_a)h_u = 0$, then

$$\begin{aligned} g(K_a)g &= \frac{1}{2e}\{g(u) \otimes [g(u)](K)g - [g(u)](K)g \otimes g(u)\} , \\ K_a &= \frac{1}{2e}\{u \otimes [g(u)](K) - [g(u)](K) \otimes u\} . \end{aligned}$$

In co-ordinate basis the necessary and sufficient condition has the forms

$$\begin{aligned} K^{ij} &= K^{ji} + \frac{1}{e}u_{\bar{l}}(u^i.K^{lj} - u^j.K^{li}) , \\ \{R_{\bar{j}nim} - R_{imjn} - \frac{1}{e}(u_{\bar{i}}R_{lmnj} - u_{\bar{j}}R_{lmni})u^l\}.u^mu^n &= 0 , \end{aligned} \quad (116)$$

where

$$R_{\bar{i}jkl} = g_{\bar{i}\bar{n}}.R^n{}_{jkl} .$$

Proposition 8. A sufficient condition for the existence of a contravariant vector field u ($g(u, u) = e \neq 0$) without rotation acceleration, induced by the curvature (i.e. with $N = 0$) is the condition

$$K_a = 0 . \quad (117)$$

Proof: From $K_a = 0$ and the form for N , $N = h_u(K_a)h_u$, it follows $N = 0$. In co-ordinate basis

$$\begin{aligned} (R^i{}_{klm}.g^{mj} - R^j{}_{klm}.g^{mi})u^ku^l &= 0 , \\ (R_{\bar{i}kjl} - R_{\bar{j}lik})u^ku^l &= 0 . \end{aligned} \quad (118)$$

$K_a = 0$ can be presented also in the form

$$[g(\xi)]([R(u, v)]u) - [g(v)]([R(u, \xi)]u) = 0 , \forall \xi, v \in T(M) .$$

In this case $M = G = g(K)g$, $I = \bar{g}[G]$.

Proposition 9. A sufficient condition for the existence of a contravariant vector field u ($g(u, u) = e \neq 0$) without rotation acceleration, induced by the curvature (i.e. with $N = 0$) is the condition

$$g(\eta, [R(\xi, v)]w) = g(\xi, [R(\eta, w)]v) , \forall \eta, \xi, v, w \in T(M) , \quad (119)$$

or in co-ordinate basis

$$R_{\bar{i}jkl} = R_{\bar{k}lij} . \quad (120)$$

Proof: Because of $R(\xi, u) = -R(u, \xi)$ and for $\eta = v$ the last expression will be identical with the sufficient condition from proposition 9.

5.2.2 Contravariant vector fields without shear acceleration ${}_sM$, induced by the curvature (${}_sM = 0$)

Proposition 10. The necessary and sufficient condition for the existence of a contravariant vector field u ($g(u, u) = e \neq 0$) without shear acceleration, induced by the curvature (i.e. with ${}_sM = 0$) is the condition

$$M = \frac{1}{n-1} \cdot I \cdot h_u = \frac{1}{n-1} \cdot \bar{g}[M] \cdot h_u . \quad (121)$$

Proof: 1. Sufficiency: From the expression for M and the definition of ${}_sM = M - \frac{1}{n-1} \cdot I \cdot h_u$ it follows ${}_sM = 0$.

2. Necessity: From ${}_sM = 0 = M - \frac{1}{n-1} \cdot I \cdot h_u$ the form of M follows.

In co-ordinate basis the necessary and sufficient condition can be written in the form

$$\begin{aligned} & \{[g_{i\bar{k}} \cdot g_{\bar{l}j} + g_{j\bar{k}} \cdot g_{\bar{l}i} - \frac{1}{e}(u_i \cdot g_{\bar{l}j} + u_j \cdot g_{\bar{l}i})u_{\bar{k}}]R^k{}_{mns}g^{sl} - \\ & - \frac{2}{n-1} \cdot R_{mn}(g_{ij} - \frac{1}{e} \cdot u_i u_j)\} u^m u^n = 0 . \end{aligned} \quad (122)$$

The condition ${}_sM = 0$ is identical with the condition for K_s :

$$K_s = \frac{1}{n-1} \cdot I \cdot h^u + \frac{1}{2e} \{u \otimes [g(u)](K) + [g(u)](K) \otimes u\} . \quad (123)$$

5.2.3 Contravariant vector fields without shear and expansion acceleration, induced by the curvature (${}_sM = 0$, $I = 0$)

Proposition 11. A sufficient condition for the existence of a contravariant vector field u ($g(u, u) = e \neq 0$) without shear and expansion acceleration, induced by the curvature (i.e. with ${}_sM = 0$, $I = 0$) is the condition

$$K_s = \frac{1}{2e} \{u \otimes [g(u)](K) + [g(u)](K) \otimes u\} . \quad (124)$$

Proof: After acting on the left and on the right side of the last expression with g

$$\begin{aligned} g(K_s)g &= \frac{1}{2e} \{g(u) \otimes [g(u)](K)g + [g(u)](K)g \otimes g(u)\} , \\ g([g(u)](K)) &= ([g(u)](K))g = [g(u)](K)g , \quad u(g) = g(u) , \end{aligned}$$

and comparing the result with the form for M ,

$$M = h_u(K_s)h_u = g(K_s)g - \frac{1}{2e} \{g(u) \otimes [g(u)](K)g + [g(u)](K)g \otimes g(u)\} ,$$

it follows that $M = 0$. Since $I = \bar{g}[M]$ it follows that $I = 0$ and ${}_sM = 0$.

Proposition 12. A sufficient condition for the existence of a contravariant vector field u ($g(u, u) = e \neq 0$) without shear and expansion acceleration, induced by the curvature (i.e. with ${}_sM = 0$, $I = 0$) is the condition

$$K_s = 0 .$$

Proof: From the condition and the form of M , $M = h_u(K_s)h_u$, it follows that $M = 0$ and therefore $I = 0$ and ${}_sM = 0$.

5.2.4 Contravariant vector fields without shear and rotation acceleration, induced by the curvature (${}_sM = 0$, $N = 0$)

Proposition 13. A sufficient condition for the existence of a contravariant vector field u ($g(u, u) = e \neq 0$) without shear and rotation acceleration, induced by the curvature (i.e. with ${}_sM = 0$, $N = 0$) is the condition

$$[R(u, \xi)]v = \frac{R}{n(n-1)}[g(v, u) \cdot \xi - g(v, \xi) \cdot u] , \quad (125)$$

$$\forall v, \xi \in T(M) , R \in C^r(M) .$$

Proof: Since v is an arbitrary contravariant vector field it can be chosen as u . Then, because of the relation

$$h_u([R(u, \xi)]u) = h_u(K)h_u(\xi) = G(\xi) , \quad (126)$$

it follows that

$$G = h_u(K)h_u = \frac{R}{n(n-1)} \cdot e \cdot h_u = G_s , G_a = h_u(K_a)h_u = 0 . \quad (127)$$

Therefore

$$M = G_s = \frac{R \cdot e}{n(n-1)} \cdot h_u , N = G_a = 0 , I = \frac{1}{n} \cdot R \cdot e , {}_sM = 0 . \quad (128)$$

In co-ordinate basis the sufficient condition can be written in the form

$$R^i{}_{jkl} = \frac{R}{n(n-1)}(g_l^i \cdot g_{jk} - g_k^i \cdot g_{jl}) \quad (129)$$

and the following relations are fulfilled

$$\begin{aligned} R_{jk} &= R^l{}_{jkl} = g_l^i \cdot R^i{}_{jkl} = \frac{1}{n} \cdot R \cdot g_{jk} , \\ R &= g^{jk} \cdot R_{jk} , \\ I &= R_{jk} \cdot u^j u^k = \frac{1}{n} \cdot R \cdot e . \end{aligned} \quad (130)$$

Proposition 14. The necessary and sufficient conditions for the existence of K in the form

$$K = \frac{1}{n-1} \cdot g[K] \cdot h^u \quad (131)$$

are the conditions

$${}_sM = 0 , K_a = 0 .$$

Proof: 1. Sufficiency: From $K_a = 0$ it follows that $K = K_s$, $N = 0$ and $M = g(K_s)g = g(K)g$. Therefore, $I = \bar{g}[M] = g[K]$. From ${}_sM = M - \frac{1}{n-1} \cdot I \cdot h_u = 0$ it follows that $M = \frac{1}{n-1} \cdot g[K] \cdot h_u = g(K)g$. From the last expression it follows the above condition for K .

2. Necessity: From the condition $K = \frac{1}{n-1} \cdot g[K] \cdot h^u$ it follows that $K = K_s$ and therefore $K_a = 0$, $N = 0$ and $M = \frac{1}{n-1} \cdot g[K] \cdot h_u$, $I = g[K]$ (because of

$h_u(h^u)h_u = h_u$, $h_u(\bar{g})h_u = h_u$). From the forms of M and I it follows that ${}_sM = 0$.

Proposition 15. A sufficient condition for the existence of a contravariant vector field u ($g(u, u) = e \neq 0$) without shear and rotation acceleration, induced by the curvature (i.e. with ${}_sM = 0$, $N = 0$) is the condition

$$K = \frac{1}{n-1} \cdot g[K] \cdot h^u .$$

Proof: Follows immediately from proposition 15.

5.2.5 Contravariant vector fields without expansion acceleration, induced by the curvature ($I = 0$)

By means of the covariant metric g and the tensor field $K(v, \xi)$ the notion contravariant Ricci tensor *Ricci* can be introduced

$$Ricci(v, \xi) = g[K(v, \xi)] , \forall v, \xi \in T(M) , \quad (132)$$

where

$$K(v, \xi) = R^i{}_{jkl} \cdot g^{lm} \cdot v^j \cdot \xi^k \cdot \partial_i \otimes \partial_m = R^\alpha{}_{\beta\gamma\kappa} \cdot g^{\kappa\delta} \cdot v^\beta \cdot \xi^\gamma \cdot e_\alpha \otimes e_\delta , \quad (133)$$

and the following relations are fulfilled

$$\begin{aligned} Ricci(e_\alpha, e_\beta) &= g[K(e_\alpha, e_\beta)] = R_{\alpha\beta} , \\ Ricci(\partial_i, \partial_j) &= g[K(\partial_i, \partial_j)] = R_{ij} , \\ Ricci(u, u) &= g[K(u, u)] = g[K] = I . \end{aligned} \quad (134)$$

Proposition 16. The necessary and sufficient condition for the existence of a contravariant vector field u ($g(u, u) = e \neq 0$) without expansion acceleration, induced by the curvature (i.e. with $I = 0$) is the condition

$$Ricci(u, u) = 0 .$$

Proof: It follows immediately from the relation $Ricci(u, u) = g[K(u, u)] = g[K] = I$.

Proposition 17. A sufficient condition for the existence of a contravariant vector field u ($g(u, u) = e \neq 0$) without expansion acceleration, induced by the curvature (i.e. with $I = 0$) is the condition

$$\begin{aligned} Ricci(e_\alpha, e_\beta) &= R_{\alpha\beta} = R^\gamma{}_{\alpha\beta\gamma} = 0 , \\ Ricci(\partial_i, \partial_j) &= R_{ij} = R^l{}_{ijl} = 0 . \end{aligned} \quad (135)$$

Proof: From $Ricci(\partial_i, \partial_j) = R_{ij} = 0$ it follows that

$$R_{ij} \cdot u^i u^j = u^i u^j \cdot Ricci(\partial_i, \partial_j) = Ricci(u, u) = I = 0 .$$

In non-co-ordinate basis the proof is analogous to that in co-ordinate basis.

The existence of contravariant vector fields with vanishing characteristics, induced by the curvature, is important for mathematical models of gravitational interactions in theories over (\bar{L}_n, g) -spaces.

6 Conclusion

The covariant and contravariant metric introduced over differentiable manifolds with contravariant and covariant affine connection allow applications for mathematical models of dynamic systems described over (\overline{L}_n, g) -spaces. On the other side different type of geometries can be considered by imposing certain additional conditions of the type of metric transport on the metric. Additional conditions determined by different "draggings along" of the metric can have physical interpretation connected with changes of the length of a vector field and with changes of the angle between two vector fields.

The introduction of contravariant and covariant projective metric corresponding to a non-isotropic (non-null) contravariant vector field allows the evolution of tensor analysis over sub-manifolds of a manifold with contravariant and covariant connection and metric and its applications for descriptions of the evolution of physical systems over (\overline{L}_n, g) -spaces.

The kinematic characteristics, connected with the introduced notions relative velocity and relative acceleration can be used for description of different dynamic systems by means of mathematical models, using differentiable manifold M with contravariant and covariant affine connection and metric as a model of space-time ($\dim M = 4$)(ETG in V_n -spaces, Einstein-Cartan theory in U_n -spaces), or as a model for the consideration of dynamic characteristics of some physical systems (theories of the type of Kaluza-Klein in V_n -spaces ($n > 4$), relativistic hydrodynamics etc.). At the same time the kinematic characteristics can be used for a more correct formulation of problems, connected with the experimental check-up of modern gravitational theories.

In the case of general relativity theory proposition 5. can be used for describing the characteristics of gravitational detectors: If test particles are considered to move in an external gravitational field ($R_{ij} = 0$), then their relative acceleration will be caused only by the curvature shear acceleration. Therefore, gravitational wave detectors have to be able to detect accelerations of the type of shear acceleration (and not of the type of expansion acceleration), if the energy-momentum tensor of the detector is neglected as a source of a gravitational field.

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A Kinematic characteristics connected with the relative acceleration and expressed in terms of the kinematic characteristics connected with the relative velocity

The deformation, shear, rotation and expansion acceleration can be expressed in terms of the shear, rotation and expansion velocity.

a) Deformation acceleration tensor A :

$$A = \frac{1}{e} h_u(a) \otimes h_u(a) + \sigma(\bar{g})\sigma + \omega(\bar{g})\omega + \frac{2}{n-1} \cdot \theta \cdot (\sigma + \omega) + \frac{1}{n-1} (\theta \cdot + \frac{\theta^2}{n-1}) h_u + \sigma(\bar{g})\omega + \omega(\bar{g})\sigma + \nabla_u \sigma + \nabla_u \omega + \frac{1}{e} h_u(a) \otimes (g(u))(2k - \nabla_u \bar{g}) h_u + \frac{1}{e} [\sigma(a) \otimes g(u) + g(u) \otimes \sigma(a)] + \frac{1}{e} [\omega(a) \otimes g(u) - g(u) \otimes \omega(a)] + h_u(\nabla_u \bar{g})\sigma + h_u(\nabla_u \bar{g})\omega , \quad (136)$$

where

$$k = \epsilon + s - (m + q) = k_0 - (m + q) , \quad k(g) \mathcal{L}_\xi u = \nabla_{\mathcal{L}_\xi u} u - T(\mathcal{L}_\xi u, u) . \quad (137)$$

In index form

$$A_{ij} = \frac{1}{e} h_{ik} \bar{a}^k a^l h_{lj} + \sigma_{ik} g^{kl} \sigma_{lj} + \omega_{ik} g^{kl} \omega_{lj} + \frac{2}{n-1} \cdot \theta \cdot \sigma_{ij} + \frac{1}{n-1} (\theta \cdot + \frac{\theta^2}{n-1}) h_{ij} + \sigma_{ij;k} u^k + \frac{1}{e} a^k [\sigma_{ik} \bar{u}_j + \sigma_{jk} \bar{u}_i + h_{k(i} h_{j)\bar{l}} \bar{u}_{\bar{n}} (2k^{nl} - g^{nl}_{;r} u^r)] + \frac{1}{2} (h_{ik} g^{kl}_{;r} u^r \sigma_{lj} + h_{jk} g^{kl}_{;r} u^r \sigma_{li}) + \frac{1}{2} (h_{ik} g^{kl}_{;r} u^r \omega_{lj} + h_{jk} g^{kl}_{;r} u^r \omega_{li}) + \sigma_{ik} g^{kl} \omega_{lj} - \sigma_{jk} g^{kl} \omega_{li} + \frac{2}{n-1} \cdot \theta \cdot \omega_{ij} + \omega_{ij;r} u^r + \frac{1}{e} a^k [\omega_{ik} \bar{u}_j - \omega_{jk} \bar{u}_i + h_{k(i} h_{j)\bar{l}} \bar{u}_{\bar{n}} (2k^{nl} - g^{nl}_{;r} u^r)] + \frac{1}{2} (h_{ik} g^{kl}_{;r} u^r \sigma_{lj} - h_{jk} g^{kl}_{;r} u^r \sigma_{li}) + \frac{1}{2} (h_{ik} g^{kl}_{;r} u^r \omega_{lj} - h_{jk} g^{kl}_{;r} u^r \omega_{li}) = D_{ij} + W_{ij} , \quad (138)$$

$$A_{(ij)} = \frac{1}{2} (A_{ij} + A_{ji}) , \quad A_{[ij]} = \frac{1}{2} (A_{ij} - A_{ji}) .$$

b) Shear acceleration tensor ${}_s D = D - \frac{1}{n-1} U \cdot h_u$:

$$D = \frac{1}{e} h_u(a) \otimes h_u(a) + \sigma(\bar{g})\sigma + \omega(\bar{g})\omega + \frac{2}{n-1} \cdot \theta \cdot \sigma + \frac{1}{n-1} (\theta \cdot + \frac{\theta^2}{n-1}) h_u + \nabla_u \sigma + \frac{1}{2e} [h_u(a) \otimes (g(u))(2k - \nabla_u \bar{g}) h_u + h_u((g(u))(2k - \nabla_u \bar{g})) \otimes h_u(a)] + \frac{1}{e} [\sigma(a) \otimes g(u) + g(u) \otimes \sigma(a)] + \frac{1}{2} [h_u(\nabla_u \bar{g})\sigma + \sigma(\nabla_u \bar{g}) h_u] + \frac{1}{2} [h_u(\nabla_u \bar{g})\omega - \omega(\nabla_u \bar{g}) h_u] . \quad (139)$$

In index form

$$D_{ij} = D_{ji} = \frac{1}{e} h_{ik} \bar{a}^k a^l h_{lj} + \sigma_{ik} g^{kl} \sigma_{lj} + \omega_{ik} g^{kl} \omega_{lj} + \frac{2}{n-1} \cdot \theta \cdot \sigma_{ij} + \frac{1}{n-1} (\theta \cdot + \frac{\theta^2}{n-1}) h_{ij} + \sigma_{ij;k} u^k + \frac{1}{e} a^k \{ \sigma_{ik} \bar{u}_j + \sigma_{jk} \bar{u}_i + h_{k(i} h_{j)\bar{l}} \bar{u}_{\bar{n}} [g^{ml} (e_{,m} - g_{rs;m} u^{\bar{r}} u^{\bar{s}} - 2T_{mr}^n u^r u_{\bar{n}}) - u_{\bar{n}} g^{nl}_{;m} u^m] \} + \frac{1}{2} (h_{ik} g^{kl}_{;m} u^m \sigma_{lj} + h_{jk} g^{kl}_{;m} u^m \sigma_{li}) + \frac{1}{2} (h_{ik} g^{kl}_{;m} u^m \omega_{lj} + h_{jk} g^{kl}_{;m} u^m \omega_{li}) . \quad (140)$$

c) Rotation acceleration tensor W

$$\begin{aligned}
&= \sigma(\bar{g})\omega + \omega(\bar{g})\sigma + \frac{2}{n-1}.\theta.\omega + \nabla_u\omega + \\
&\quad + \frac{1}{e}[\omega(a) \otimes g(u) - g(u) \otimes \omega(a)] + \\
&+ \frac{1}{2e}[h_u(a) \otimes (g(u))(2k - \nabla_u\bar{g})h_u - h_u((g(u))(2k - \nabla_u\bar{g})) \otimes h_u(a)] + \\
&\quad + \frac{1}{2}[h_u(\nabla_u\bar{g})\sigma - \sigma(\nabla_u\bar{g})h_u] + \frac{1}{2}[h_u(\nabla_u\bar{g})\omega + \omega(\nabla_u\bar{g})h_u] .
\end{aligned} \tag{141}$$

In index form

$$\begin{aligned}
W_{ij} = -W_{ji} = &\sigma_{ik}g^{\bar{k}l}\omega_{lj} - \sigma_{jk}g^{\bar{k}l}\omega_{li} + \frac{2}{n-1}.\theta.\omega_{ij} + \omega_{ij;k}u^k + \\
&+ \frac{1}{e}a^k\{\omega_{i\bar{k}}u_j - \omega_{j\bar{k}}u_i + h_{\bar{k}[i}h_{j]\bar{l}}[g^{ml}(e_{,m} - g_{rs;m}u^{\bar{r}}u^{\bar{s}} - \\
&\quad - 2T_{mr}^nu^ru_{\bar{n}}) - u_{\bar{n}}g^{nl}_{;m}u^m]\} + \\
&+ \frac{1}{2}(h_{ik}g^{\bar{k}l}_{;m}u^m\sigma_{lj} - h_{jk}g^{\bar{k}l}_{;m}u^m\sigma_{li}) + \\
&+ \frac{1}{2}(h_{i\bar{k}}g^{kl}_{;m}u^m\omega_{\bar{l}j} - h_{j\bar{k}}g^{kl}_{;m}u^m\omega_{\bar{l}i}) .
\end{aligned} \tag{142}$$

d) Expansion acceleration U

$$\begin{aligned}
U = &\frac{1}{e}.g(a, a) + \bar{g}[\sigma(\bar{g})\sigma] + \bar{g}[\omega(\bar{g})\omega] + \theta + \frac{1}{n-1}.\theta^2 + \\
&+ \frac{1}{e}[2g(u, \nabla_a u) - 2g(u, T(a, u)) + (\nabla_u g)(a, u)] \\
&\quad - \frac{1}{e^2}.g(u, a).[3g(u, a) + (\nabla_u g)(u, u)] .
\end{aligned} \tag{143}$$

In index form

$$\begin{aligned}
U = &\frac{1}{e}.g_{ij}a^ia^j + g^{\bar{i}j}g^{\bar{k}l}\sigma_{ik}\sigma_{jl} - g^{\bar{i}j}g^{\bar{k}l}\omega_{ik}\omega_{jl} + \theta + \frac{1}{n-1}.\theta^2 + \\
&+ \frac{1}{e}.g_{kl}a^k[g^{ml}(e_{,m} - g_{rs;m}u^{\bar{r}}u^{\bar{s}} - 2T_{mr}^nu^ru_{\bar{n}}) - u_{\bar{n}}g^{nl}_{;m}u^m] - \\
&+ \frac{1}{e^2}[\frac{3}{4}(e_{,k}u^k)^2 - (e_{,l}u^l)g_{ij;k}u^ku^iu^j + \frac{1}{4}(g_{ij;k}u^ku^iu^j)^2] .
\end{aligned} \tag{144}$$

e) Torsion-free and curvature-free shear acceleration tensor ${}_sFD_0$

$$\begin{aligned}
{}_sFD_0 = &{}_FD_0 - \frac{1}{n-1}._FU_0.h_u \\
{}_FD_0 = &h_u(b_s)h_u , \quad {}_FU_0 = g[b] - \frac{1}{e}.g(u, \nabla_u a) .
\end{aligned} \tag{145}$$

In index form

$$({}_FD_0)_{ij} = ({}_FD_0)_{ji} = \frac{1}{2}.h_{i\bar{k}}(a^k_{;n}g^{nl} + a^l_{;n}g^{nk})h_{\bar{l}j} , \tag{146}$$

$$\begin{aligned}
{}_FU_0 = &a^k_{;k} - \frac{1}{e}.g_{kl}u^ka^l_{;m}u^m = \\
= &a^k_{;k} - \frac{1}{e}[(g_{kl}u^ka^l)_{;m}u^m - g_{kl;m}u^mu^{\bar{k}}a^{\bar{l}} - g_{\bar{k}l}a^ka^l] .
\end{aligned} \tag{147}$$

f) Torsion-free and curvature-free rotation acceleration tensor ${}_FW_0$

$${}_FW_0 = h_u(b_a)h_u . \tag{148}$$

In index form

$$({}_FW_0)_{ij} = -({}_FW_0)_{ji} = \frac{1}{2}.h_{i\bar{k}}(a^k_{;n}g^{nl} - a^l_{;n}g^{nk})h_{\bar{l}j} . \tag{149}$$

- g) Torsion-free and curvature-free expansion acceleration ${}_F U_0$ (s. e)).
h) Curvature-free shear acceleration tensor ${}_s D_0 = D_0 - \frac{1}{n-1} \cdot U_0 \cdot h_u$

$$\begin{aligned}
D_0 = & h_u(b_s)h_u - \frac{1}{2}[{}_s P(\bar{g})\sigma + \sigma(\bar{g}){}_s P] - \frac{1}{2}[Q(\bar{g})\omega + \omega(\bar{g})Q] - \\
& - \frac{1}{n-1}(\theta_1 \cdot \sigma + \theta \cdot {}_s P) - \frac{1}{n-1}(\theta \cdot + \frac{1}{n-1} \cdot \theta_1 \cdot \theta)h_u - \nabla_u({}_s P) - \\
& - \frac{1}{2}[{}_s P(\bar{g})\omega - \omega(\bar{g}){}_s P] - \frac{1}{2}[Q(\bar{g})\sigma - \sigma(\bar{g})Q] - \\
& - \frac{1}{2e}[h_u(a) \otimes (g(u))(m+q)h_u + h_u((g(u))(m+q)) \otimes h_u(a) - \\
& - \frac{1}{e}[{}_s P(a) \otimes g(u) + g(u) \otimes {}_s P(a)] - \\
& - \frac{1}{2}[h_u(\nabla_u \bar{g}){}_s P + {}_s P(\nabla_u \bar{g})h_u] - \frac{1}{2}[h_u(\nabla_u \bar{g})Q - Q(\nabla_u \bar{g})h_u] .
\end{aligned} \tag{150}$$

In index form

$$\begin{aligned}
(D_0)_{ij} = (D_0)_{ji} = & h_{\bar{k}(i}h_{j)\bar{l}}a^k{}_{;m}g^{ml} - {}_s P_{k(i}\sigma_{j)l}g^{\bar{k}\bar{l}} - Q_{k(i}\omega_{j)l}g^{\bar{k}\bar{l}} - \\
& - \frac{1}{n-1}(\theta_1 \cdot \sigma_{ij} + \theta \cdot {}_s P_{ij}) - \frac{1}{n-1}(\theta_1 + \frac{1}{n-1} \cdot \theta_1 \cdot \theta)h_{ij} - \\
& - {}_s P_{ij;m}u^m + {}_s P_{k(i}\omega_{j)l}g^{\bar{k}\bar{l}} + Q_{k(i}\sigma_{j)l}g^{\bar{k}\bar{l}} - \\
& - \frac{1}{e} \cdot a^k[{}_s P_{ik}u_j + {}_s P_{jk}u_i + h_{\bar{k}(i}h_{j)\bar{l}}u_{\bar{m}}T_{mr}^n u^r g^{ml}] - \\
& - {}_s P_{\bar{k}(i}h_{j)\bar{l}}g^{kl}{}_{;m}u^m - Q_{\bar{k}(i}h_{j)\bar{l}}g^{kl}{}_{;m}u^m .
\end{aligned} \tag{151}$$

- i) Curvature-free rotation acceleration tensor W_0

$$\begin{aligned}
W_0 = & h_u(b_a)h_u - \frac{1}{2}[{}_s P(\bar{g})\sigma - \sigma(\bar{g}){}_s P] - \frac{1}{2}[Q(\bar{g})\omega - \omega(\bar{g})Q] - \\
& - \frac{1}{n-1}(\theta_1 \cdot \omega + \theta \cdot Q) - \nabla_u Q - \frac{1}{2}[{}_s P(\bar{g})\omega + \omega(\bar{g}){}_s P] - \\
& - \frac{1}{2}[Q(\bar{g})\sigma + \sigma(\bar{g})Q] - \frac{1}{e}[Q(a) \otimes g(u) - g(u) \otimes Q(a)] - \\
& - \frac{1}{2e}[h_u(a) \otimes (g(u))(m+q)h_u - h_u((g(u))(m+q)) \otimes h_u(a)] - \\
& - \frac{1}{2}[h_u(\nabla_u \bar{g}){}_s P - {}_s P(\nabla_u \bar{g})h_u] - \frac{1}{2}[h_u(\nabla_u \bar{g})Q + Q(\nabla_u \bar{g})h_u] .
\end{aligned} \tag{152}$$

In index form

$$\begin{aligned}
(W_0)_{ij} = -(W_0)_{ji} = & h_{\bar{k}[i}h_{j]\bar{l}}a^k{}_{;m}g^{ml} - {}_s P_{\bar{k}[i}\sigma_{j]\bar{l}}g^{kl} - Q_{\bar{k}[i}\omega_{j]\bar{l}}g^{kl} - \\
& - \frac{1}{n-1}(\theta_1 \cdot \omega_{ij} + \theta \cdot Q_{ij}) - Q_{ij;m}u^m + \\
& + {}_s P_{k[i}\omega_{j]l}g^{\bar{k}\bar{l}} + Q_{k[i}\sigma_{j]l}g^{\bar{k}\bar{l}} - \\
& - \frac{1}{e} \cdot a^k(Q_{ik}u_j - Q_{jk}u_i + h_{\bar{k}[i}h_{j]\bar{l}}u_{\bar{m}}T_{mr}^n u^r g^{ml}) + \\
& + {}_s P_{\bar{k}[i}h_{j]\bar{l}}g^{kl}{}_{;m}u^m + Q_{\bar{k}[i}h_{j]\bar{l}}g^{kl}{}_{;m}u^m .
\end{aligned} \tag{153}$$

- j) Curvature-free expansion acceleration U_0

$$\begin{aligned}
U_0 = & g[b] - \bar{g}[{}_s P(\bar{g})\sigma] - \bar{g}[Q(\bar{g})\omega] - \theta_1 - \frac{1}{n-1} \cdot \theta_1 \cdot \theta - \\
& - \frac{1}{e}[g(u, T(a, u)) + g(u, \nabla_u a)] .
\end{aligned} \tag{154}$$

In index form

$$\begin{aligned}
U_0 = & a^k{}_{;k} - g^{\bar{i}\bar{j}} \cdot {}_s P_{ik}g^{\bar{k}\bar{l}}\sigma_{lj} - g^{\bar{i}\bar{j}}Q_{ik}g^{\bar{k}\bar{l}}\omega_{lj} - \theta_1 - \frac{1}{n-1} \cdot \theta_1 \cdot \theta - \\
& - \frac{1}{e}[a^k(u_{\bar{n}}T_{km}^n u^m - 2g_{\bar{k}\bar{m};l}u^l u^m - g_{\bar{k}\bar{l}}a^l) + \\
& + \frac{1}{2}(e_{,k}u^k)_{,l}u^l - \frac{1}{2}(g_{mn;r}u^r)_{;s}u^s u^{\bar{m}}u^{\bar{n}}] .
\end{aligned} \tag{155}$$

- k) Shear acceleration tensor, induced by the torsion, ${}_s T D_0$

$${}_s T D_0 = {}_T D_0 - \frac{1}{n-1} \cdot {}_T U_0 \cdot h_u$$

$$\begin{aligned}
{}_TD_0 = & \frac{1}{2}[{}_sP(\bar{g})\sigma + \sigma(\bar{g}){}_sP] + \frac{1}{2}[Q(\bar{g})\omega + \omega(\bar{g})Q] + \\
& + \frac{1}{n-1}(\theta_1.\sigma + \theta.{}_sP) + \frac{1}{n-1}(\theta_1 + \frac{1}{n-1}.\theta_1.\theta)h_u + \nabla_u({}_sP) + \\
& + \frac{1}{2}[{}_sP(\bar{g})\omega - \omega(\bar{g}){}_sP] + \frac{1}{2}[Q(\bar{g})\sigma - \sigma(\bar{g})Q] + \\
& + \frac{1}{2e}[h_u(a) \otimes (g(u))(m+q)h_u + h_u((g(u))(m+q)) \otimes h_u(a)] + \\
& + \frac{1}{e}[{}_sP(a) \otimes g(u) + g(u) \otimes {}_sP(a)] + \\
& + \frac{1}{2}[h_u(\nabla_u\bar{g}){}_sP + {}_sP(\nabla_u\bar{g})h_u] + \frac{1}{2}[h_u(\nabla_u g)Q - Q(\nabla_u g)h_u] .
\end{aligned} \tag{156}$$

In index form

$$({}_TD_0)_{ij} = ({}_FD_0)_{ij} - (D_0)_{ij} . \tag{157}$$

l) Expansion acceleration, induced by the torsion, ${}_TU_0$

$${}_TU_0 = \bar{g}[{}_sP(\bar{g})\sigma] + \bar{g}[Q(\bar{g})\omega] + \theta_1 + \frac{1}{n-1}.\theta_1.\theta + \frac{1}{e}.g(u, T(a, u)) . \tag{158}$$

In index form

$${}_TU_0 = {}_FU_0 - U_0 .$$

m) Rotation acceleration tensor, induced by the torsion, ${}_TW_0$

$$\begin{aligned}
{}_TW_0 = & \frac{1}{2}[{}_sP(\bar{g})\sigma - \sigma(\bar{g}){}_sP] + \frac{1}{2}[Q(\bar{g})\omega - \omega(\bar{g})Q] + \\
& + \frac{1}{n-1}(\theta_1.\omega + \theta.Q) + \nabla_u Q + \frac{1}{2}[{}_sP(\bar{g})\omega + \omega(\bar{g}){}_sP] + \\
& + \frac{1}{2}[Q(\bar{g})\sigma + \sigma(\bar{g})Q] + \\
& + \frac{1}{2e}[h_u(a) \otimes (g(u))(m+q)h_u - h_u((g(u))(m+q)) \otimes h_u(a)] + \\
& + \frac{1}{e}[Q(a) \otimes g(u) - g(u) \otimes Q(a)] + \\
& + \frac{1}{2}[h_u(\nabla_u\bar{g}){}_sP - {}_sP(\nabla_u\bar{g})h_u] + \frac{1}{2}[h_u(\nabla_u\bar{g})Q + Q(\nabla_u\bar{g})h_u] .
\end{aligned} \tag{159}$$

In index form

$$({}_TW_0)_{ij} = ({}_FW_0)_{ij} - (W_0)_{ij} . \tag{160}$$

n) Shear acceleration tensor, induced by the curvature, ${}_sM = M - \frac{1}{n-1}.I.h_u$

$$\begin{aligned}
M = & \frac{1}{e}.h_u(a) \otimes h_u(a) + \frac{1}{2}[{}_sE(\bar{g})\sigma + \sigma(\bar{g}){}_sE] + \frac{1}{2}[S(\bar{g})\omega + \omega(\bar{g})S] + \\
& + \frac{1}{n-1}(\theta_o.\sigma + \theta.{}_sE) + \frac{1}{n-1}(\theta_o + \frac{1}{n-1}.\theta_o.\theta)h_u + \nabla_u({}_sE) + \\
& + \frac{1}{2}[{}_sE(\bar{g})\omega - \omega(\bar{g}){}_sE] + \frac{1}{2}[S(\bar{g})\sigma - \sigma(\bar{g})S] + \\
& + \frac{1}{2e}[h_u(a) \otimes (g(u))(k_0 + k - \nabla_u\bar{g})h_u + h_u((g(u))(k_0 + k - \nabla_u\bar{g})) \otimes h_u(a)] + \\
& + \frac{1}{e}[{}_sE(a) \otimes g(u) + g(u) \otimes {}_sE(a)] + \\
& + \frac{1}{2}[h_u(\nabla_u\bar{g}){}_sE + {}_sE(\nabla_u\bar{g})h_u] + \frac{1}{2}[h_u(\nabla_u\bar{g})S - S(\nabla_u\bar{g})h_u] - \\
& - h_u(b_s)h_u .
\end{aligned} \tag{161}$$

In index form

$$\begin{aligned}
M_{ij} = M_{ji} = & \frac{1}{e}.h_{i\bar{k}}a^ka^lh_{\bar{l}j} + {}_sE_{k(i}\sigma_{j)l}g^{\bar{k}l} + S_{k(i}\omega_{j)l}g^{\bar{k}l} + \\
& + \frac{1}{n-1}(\theta_o.\sigma_{ij} + \theta.{}_sE_{ij}) + \frac{1}{n-1}(\theta_o + \frac{1}{n-1}.\theta_o.\theta)h_{ij} - \\
& - {}_sE_{k(i}\omega_{j)l}g^{\bar{k}l} - S_{k(i}\sigma_{j)l}g^{\bar{k}l} + {}_sE_{ij;k}u^k + \\
& + \frac{1}{e}.a^k[{}_sE_{i\bar{k}}u_j + {}_sE_{j\bar{k}}u_i + h_{\bar{k}(i}h_{j)\bar{l}}\bar{g}^{ml}(e_{,m} - u_{\bar{n}}T_{mr}^nu^r - \\
& - g_{rs;m}u^ru^s + g_{\bar{m}\bar{r};s}u^su^r)] + \\
& + {}_sE_{\bar{k}(i}h_{j)\bar{l}}\bar{g}^{kl}{}_{;s}u^s + S_{\bar{k}(i}h_{j)\bar{l}}\bar{g}^{kl}{}_{;s}u^s - \\
& - h_{\bar{k}(i}h_{j)\bar{l}}\bar{a}^k{}_{;m}g^{ml} .
\end{aligned} \tag{162}$$

o) Expansion acceleration, induced by the curvature, I

$$I = -g[b] + \bar{g}[{}_s E(\bar{g})\sigma] + \bar{g}[S(\bar{g})\omega] + \theta_o + \frac{1}{n-1}.\theta_o.\theta + \frac{1}{e}[2g(u, \nabla_a u) - g(u, T(a, u)) + u(g(u, a))] - \frac{1}{e^2}.g(u, a)[3g(u, a) + (\nabla_u g)(u, u)] . \quad (163)$$

In index form

$$I = R_{ij}u^i u^j = -a^j{}_{;j} + g^{ij}g^{kl}{}_s E_{ik}\sigma_{lj} + g^{ij}g^{kl}S_{ik}\omega_{lj} + \theta_o + \frac{1}{n-1}.\theta_o.\theta + \frac{1}{e}[a^k(e_{,k} - u_{\bar{n}}T_{km}^n u^m - g_{mn;k}u^{\bar{m}}u^{\bar{n}} - g_{\bar{k}\bar{m};s}u^s u^m) + \frac{1}{2}(u^k e_{,k})_{,l}u^l - \frac{1}{2}.(g_{mn;r}u^r)_{;s}u^s u^{\bar{m}}u^{\bar{n}}] - \frac{1}{e^2}[\frac{3}{4}(e_{,k}u^k)^2 - (e_{,k}u^k)g_{mn;r}u^r u^{\bar{m}}u^{\bar{n}} + \frac{1}{4}(g_{mn;r}u^r u^{\bar{m}}u^{\bar{n}})^2] . \quad (164)$$

p) Rotation expansion tensor, induced by the curvature, N

$$N = \frac{1}{2}[{}_s E(\bar{g})\sigma - \sigma(\bar{g}){}_s E] + \frac{1}{2}[S(\bar{g})\omega - \omega(\bar{g})S] + \frac{1}{n-1}(\theta_o.\omega + \theta_o.S) + \nabla_u S + \frac{1}{2}[{}_s E(\bar{g})\omega + \omega(\bar{g}){}_s E] + \frac{1}{2}[S(\bar{g})\sigma + \sigma(\bar{g})S] + \frac{1}{2e}[h_u(a) \otimes (g(u))(k_0 + k - \nabla_u \bar{g})h_u - h_u((g(u))(k_0 + k - \nabla_u \bar{g})) \otimes h_u(a)] + \frac{1}{e}[S(a) \otimes g(u) - g(u) \otimes S(a)] + \frac{1}{2}[h_u(\nabla_u \bar{g}){}_s E - {}_s E(\nabla_u \bar{g})h_u] + \frac{1}{2}[h_u(\nabla_u \bar{g})S + S(\nabla_u \bar{g})h_u] - h_u(b_a)h_u . \quad (165)$$

In index form

$$N_{ij} = -N_{ji} = {}_s E_{k[i}\sigma_{j]l}g^{kl} + S_{k[i}\omega_{j]l}g^{kl} + \frac{1}{n-1}(\theta_o.\omega_{ij} + \theta_o.S_{ij}) - {}_s E_{k[i}\omega_{j]l}g^{kl} - S_{k[i}\sigma_{j]l}g^{kl} + S_{ij;k}u^k - h_{\bar{k}[i}h_{j]\bar{l}}a^{\bar{k}}{}_{;m}g^{ml} + \frac{1}{e}.a^k[S_{ik}u_j - S_{jk}u_i + h_{\bar{k}[i}h_{j]\bar{l}}g^{ml}(e_{,m} - u_{\bar{n}}T_{mr}^n u^r - g_{rs;m}u^{\bar{r}}u^{\bar{s}} + g_{\bar{m}\bar{r};s}u^s u^r)] - {}_s E_{\bar{k}[i}h_{j]\bar{l}}g^{kl}{}_{;s}u^s - S_{\bar{k}[i}h_{j]\bar{l}}g^{kl}{}_{;s}u^s . \quad (166)$$

A Table 1. Kinematic characteristics connected with the notions relative velocity and relative acceleration

A.1 Kinematic characteristics connected with the relative velocity:

1. Relative position vector field
(relative position vector) $\xi_{\perp} = \bar{g}(h_u(\xi))$
2. Relative velocity ${}_{rel}v = \bar{g}(h_u(\nabla_u \xi))$
3. Deformation velocity tensor
(deformation velocity, deformation) .. $d = d_0 - d_1 = \sigma + \omega + \frac{1}{n-1}.\theta.h_u$
4. Torsion-free deformation velocity tensor
(torsion-free deformation velocity, torsion-free deformation)
..... $d_0 = {}_s E + S + \frac{1}{n-1}.\theta_o.h_u$

5. Deformation velocity tensor induced by the torsion
(torsion deformation velocity, torsion deformation)
..... $d_1 = {}_s P + Q + \frac{1}{n-1} \cdot \theta_1 \cdot h_u$
6. Shear velocity tensor
(shear velocity, shear) $\sigma = {}_s E - {}_s P$
7. Torsion-free shear velocity tensor
(torsion shear velocity, torsion shear) ${}_s E = E - \frac{1}{n-1} \cdot \theta_o \cdot h_u$
8. Shear velocity tensor induced by the torsion
(torsion shear velocity tensor, torsion shear velocity, torsion shear)
..... ${}_s P = P - \frac{1}{n-1} \cdot \theta_1 \cdot h_u$
9. Rotation velocity tensor
(rotation velocity, rotation) $\omega = S - Q$
10. Torsion-free rotation velocity tensor
(torsion-free rotation velocity, torsion-free rotation)
..... $S = h_u(s)h_u$
11. Rotation velocity tensor induced by the torsion
(torsion rotation velocity, torsion rotation) $Q = h_u(q)h_u$
12. Expansion velocity
(expansion) $\theta = \theta_o - \theta_1$
13. Torsion-free expansion velocity
(torsion-free expansion) $\theta_o = \bar{g}[E]$
14. Expansion velocity induced by the torsion
(torsion expansion velocity, torsion expansion) $\theta_1 = \bar{g}[P]$.

A.2 Kinematic characteristics connected with the relative acceleration:

1. Acceleration $a = \nabla_u u$
2. Relative acceleration ${}_{rel} a = \bar{g}(h_u(\nabla_u \nabla_u \xi))$
3. Deformation acceleration tensor
(deformation acceleration) $A = {}_s D + W + \frac{1}{n-1} \cdot U \cdot h_u$
..... $A = A_0 + G$
..... $A = {}_F A_0 - {}_T A_0 + G$
4. Torsion-free and curvature-free deformation acceleration tensor
(torsion-free and curvature-free deformation acceleration)
..... ${}_F A_0 = {}_s F D_0 + {}_F W_0 + \frac{1}{n-1} \cdot {}_F U_0 \cdot h_u$
- 4.a. Curvature-free deformation acceleration tensor
(curvature-free deformation acceleration) ... $A_0 = {}_s D_0 + W_0 + \frac{1}{n-1} \cdot U_0 \cdot h_u$
5. Deformation acceleration tensor induced by the torsion
(torsion deformation acceleration tensor, torsion deformation acceleration)
..... ${}_T A_0 = {}_s T D_0 + {}_T W_0 + \frac{1}{n-1} \cdot {}_T U_0 \cdot h_u$
- 5.a. Deformation acceleration tensor induced by the curvature
(curvature deformation acceleration tensor, curvature deformation acceleration)
..... $G = {}_s M + N + \frac{1}{n-1} \cdot I \cdot h_u$

6. Shear acceleration tensor	
(shear acceleration)	${}_s D = D - \frac{1}{n-1} \cdot U \cdot h_u$
.....	${}_s D = {}_s D_0 + {}_s M$
.....	${}_s D = {}_{sF} D_0 - {}_{sT} D_0 + {}_s M$
7. Torsion-free and curvature-free shear acceleration tensor	
(torsion-free and curvature-free shear acceleration)	
.....	${}_{sF} D_0 = {}_F D_0 - \frac{1}{n-1} \cdot {}_F U_0 \cdot h_u$
7.a. Curvature-free shear acceleration tensor	
(curvature-free shear acceleration)	${}_s D_0 = D_0 - \frac{1}{n-1} \cdot U_0 \cdot h_u$
.....	${}_s D_0 = {}_{sF} D_0 - {}_{sT} D_0$
8. Shear acceleration tensor induced by the torsion	
(torsion shear acceleration tensor, torsion shear acceleration)	
.....	${}_{sT} D_0 = {}_T D_0 - \frac{1}{n-1} \cdot {}_T U_0 \cdot h_u$
8.a. Shear acceleration tensor induced by the curvature	
(curvature shear acceleration tensor, curvature shear acceleration)	
.....	${}_s M = M - \frac{1}{n-1} \cdot I \cdot h_u$
9. Rotation acceleration tensor	
(rotation acceleration)	$W = W_0 + N$
.....	$W = {}_F W_0 - {}_T W_0 + N$
10. Torsion-free and curvature-free rotation acceleration tensor	
(torsion-free and curvature-free rotation acceleration)	
.....	${}_F W_0 = h_u(b_a)h_u$
10.a. Curvature-free rotation acceleration tensor	
(curvature-free rotation acceleration)	$W_0 = W - N$
.....	$W_0 = {}_F W_0 - {}_T W_0$
11. Rotation acceleration tensor induced by the torsion	
(torsion rotation acceleration tensor, torsion rotation acceleration)	
.....	${}_T W_0 = {}_F W_0 - W_0$
11.a. Rotation acceleration tensor induced by the curvature	
(curvature rotation acceleration tensor, curvature rotation acceleration)	
.....	$N = h_u(K_a)h_u$
12. Expansion acceleration	$U = U_0 + I$
.....	$U = {}_F U_0 - {}_T U_0 + I$
13. Torsion-free and curvature-free expansion acceleration	
.....	${}_F U_0 = \bar{g}[_F D_0]$
13.a. Curvature-free expansion acceleration ...	$U_0 = \bar{g}[D_0]$
.....	$U_0 = {}_F U_0 - {}_T U_0$
14. Expansion acceleration induced by the torsion	
(torsion expansion acceleration)	$U_0 = \bar{g}[_T D_0]$
14.a. Expansion acceleration induced by the curvature	
(curvature expansion acceleration)	$I = \bar{g}[M] = \bar{g}[G]$

B Table 2. Classification of non-isotropic auto-parallel vector fields on the basis of the kinematic characteristics connected with the relative velocity and relative acceleration

B.1 Classification on the basis of kinematic characteristics connected with the relative velocity

The following conditions, connected with the relative velocity, can characterize the vector fields over manifolds with affine connection and metric:

1. $\sigma = 0$.
2. $\omega = 0$.
3. $\theta = 0$.
4. $\sigma = 0, \omega = 0$.
5. $\sigma = 0, \theta = 0$.
6. $\omega = 0, \theta = 0$.
7. $\sigma = 0, \omega = 0, \theta = 0$.
8. ${}_sE = 0$.
9. $S = 0$.
10. $\theta_o = 0$.
11. ${}_sE = 0, S = 0$.
12. ${}_sE = 0, \theta_o = 0$.
13. $S = 0, \theta_o = 0$.
14. ${}_sE = 0, S = 0, \theta_o = 0$.
15. ${}_sP = 0$.
16. $Q = 0$.
17. $\theta_1 = 0$.
18. ${}_sP = 0, Q = 0$.
19. ${}_sP = 0, \theta_1 = 0$.
20. $Q = 0, \theta_1 = 0$.
21. ${}_sP = 0, Q = 0, \theta_1 = 0$.

B.2 Classification on the basis of the kinematic characteristics connected with the relative acceleration

The following conditions, connected with the relative acceleration, can characterize the vector fields over manifolds with affine connection and metric:

1. ${}_sD = 0$.
2. $W = 0$.
3. $U = 0$.
4. ${}_sD = 0, W = 0$.
5. ${}_sD = 0, U = 0$.
6. $W = 0, U = 0$.
7. ${}_sD = 0, W = 0, U = 0$.

8. ${}_sM = 0$.
9. $N = 0$.
10. $I = 0$.
11. ${}_sM = 0, N = 0$.
12. ${}_sM = 0, I = 0$.
13. $N = 0, I = 0$.
14. ${}_sM = 0, N = 0, I = 0$.
15. ${}_sT D_0 = 0$.
16. ${}_TW_0 = 0$.
17. ${}_TU_0 = 0$.
18. ${}_sT D_0 = 0, {}_TW_0 = 0$.
19. ${}_sT D_0 = 0, {}_TU_0 = 0$.
20. ${}_TW_0 = 0, {}_TU_0 = 0$.
21. ${}_sT D_0 = 0, {}_TW_0 = 0, {}_TU_0 = 0$.